Modelling the Right-Tail Conditional Expectation and Variance of Various Philippine Stocks Return using the Class of Beta Generalized Pareto Distribution

Angelo E. Marasigan
University of the Philippines Los Baños

A risk measure such as the *value-at-risk* (VaR) is commonly used by financial institution for capital management and calculation of amount of risk exposure against a loss. The incoherence of VaR leads to the calculation of *conditional tail expectation* (CTE) for remedy. In this study, formulas for the CTE and the *conditional tail variance* (CTV) under the *class of beta generalized Pareto* (bgP) distribution were derived. bgP is used to model the distribution of different scenarios of return of various Philippine stock indices using maximum likelihood estimation due to simulated annealing method with R software, and further used to compute the CTE and CTV of the data of returns according to the specified model. To determine the performance of bgP to model financial data sets, a comparison with its generating distributions which are the (generalized) beta and Pareto models were done. Finally, the method of historical simulation was also done and used to compute the corresponding VaR, CTE, and CTV for comparison to the above method of calculations.

Keywords. risk measure, Beta generalized Pareto distribution, value-at-risk, conditional tail expectation, conditional tail variance, heavy-tailed

1. Introduction

In different areas of applied science such as hydrology, engineering science, and applied mathematics in finance and actuarial science, the study and application of risk measure are crucial. This measure associates a number that quantifies the intensity of loss. Specifically, for a random variable , risk measure is a mapping of to a value specifying the amount of risk exposure associated with . Commonly, the *value-at-risk* (VaR), a risk measure, is used by practitioners to quantify the minimum amount such that only at a certain small probability that the random variable exceeds that amount. But VaR fails to be a coherent measure. According to Artzner et al. (1999) a coherent risk measure must satisfy four properties, namely, monotonicity, subadditivity, positive homogeniety, and translation invariance. It is by these properties that a risk measure must satisfy to become desirable to use. So, the *conditional tail expectation* (CTE) was formulated to remedy the incoherence of VaR. This measures the average of the random variable given that it exceeds VaR.

It is evident from various literatures that CTE has been extensively used to analyze different distribution models. For instance, Cai and Li (2005) developed an explicit formula of the CTE of convolutions and extreme values of dependent risks under a multivariate phase-type distribution. In the paper of Kim and Kim (2019), closed-form formula of CTE for univariate and multivariate class of normal mean-variance mixture were derived. An innovative expression for the CTE under the class of skew generalized hyperbolic family of distribution was formulated by Ignatieva and Landsman (2019) and further illustrated using the data of various stocks. Yang et al. (2015) considered studying the CTE of a weighted sum of regularly varying random variables, with each variable having dependent structure over their corresponding weights.

Another risk measure which is important in capturing the tail variability of a random variable is the *conditional tail variance* (CTV). Furman and Landsman (2006) argued that it is important to consider the tail variability due to its property of providing relevant information about the characteristic of a particular risk random variable. Jiang and Yang (2011) studied portfolio optimization using CTV and derived an explicit solution when the distribution of the portfolio is under a multivariate student-t distribution. Expressions for the CTV was also derived by Landsman et al. (2013) under the assumption of log-elliptical classes of risk. Under the assumption of generalized Laplace distribution and its mixture, CTV formulas were derived and, consequently, used for portfolio optimization by Jiang et al. (2016).

Apart from studying the tail characteristics of a random variable, it is also worthy to note its probability distribution. Development of distribution models has gained researchers interest because of their wide applicability and extension to analyze various financial, actuarial, climatic, and hydrological data sets. These developments are important in a way that it enhances the shape, location, and scale structure of a given classical distribution model to accurately fit a data set. Benkhelifa (2017) introduced a five-parameter model which is the beta Generalized Gompertz distribution and applied the model to a lifetime data of 50 devices. The same data set was used by Bagheri et al. (2016) to apply and study the flexibility and applicability of their proposed five-parameter lifetime distribution model which is the generalized modified Weibull power series distribution. An extension of the three parameter Kappa distribution which is the five-parameter Kumaraswamy generalized Kappa distribution was developed by Nawaz et al. (2018) and used stream flow amount data set to illustrate the model. The distribution model was said to be more flexible in modeling highly skewed data. A new distribution model, which is the generalized log-Moyal distribution which gained importance in actuarial risk modeling, was introduced and the distributional properties was further developed by Bhati and Ravi (2018). In the study of Arslan et al. (2017), they proposed to use the generalized Lindley and power Lindley distributions as suitable alternatives to model wind speed data instead of Weibull distribution and showed that the two former distributions dominated the latter according to various goodness-of-fit criterion such as the root mean square error, Akaike information criterion and so on. The paper of Chen and Singh (2018) studied modeling of hydrometeorological extremes by means of deriving distribution models using entropy theory. By combining the gamma and generalized normal distributions, Cordeiro et al. (2019) formed the gamma generalized normal distribution and used it to model synthetic aperture radar imagery.

This paper extended the result of Mahmoudi (2011) by providing a closed-form formulae for the CTE and CTV of the random variable developed in the said paper. After this, application to real data was done. In his study, the beta generalized Pareto distribution was introduced and argued that it is applicable to extreme value data due to its long tail feature. Application to real-time data on Philippine stock price indices was done due to known long tail behavior of financial data.

The paper is organized as follows: section 2 presents the beta generalized Pareto distributions and provides the distributional properties such as the k^{th} raw moment and the quantile function. The risk measures: VaR, CTE and CTV, are also introduced in this section. In section 3, derivation of the CTE and CTV for the beta generalized Pareto random variables are shown. Application to Philippine stock data sets is presented also, where the beta generalized distribution model is compared with its generating models which are the generalized beta and generalized pareto distributions. Moreover, comparison of CTE to total expectation and CTV to total variance are done. Further comparison to risk measures when using the method of historical simulation to compute VaR, CTE, and CTV is also done in this section. The final section concludes the paper.

2. Preliminaries

The first subsection is devoted to introducing the distribution model of generalized Pareto type with beta generator, due to their wide application in modeling various data which are heavy tailed in distribution. The next subsection introduces VaR, CTE, and CTV as risk measures. Before we proceed, the following notations are needed:

Beta Function, B, and the Incomplete Beta Function, B_1:

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt,\ B_1(x;a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt.$$

Beta Cumulative Distribution, with parameters a and b, $\overline{B}(;a,b)$:

$$\overline{B}(x;a,b)=\int_0^x \frac{B_1(t;a,b)}{B(a,b)}dt.$$

2.1. The class of beta generalized pareto distribution

The beta generalized Pareto (bgP) distribution, as proposed by Mahmoudi (2011), is a five-parameter distribution model which is an extension of the generalized Pareto distribution with beta distribution generator. In his paper, the performance and superiority of bgP over its submodels such as generalized Pareto, beta Pareto, Pareto, and the three-parameter Weibull distribution were provided and showed its dominance over those models.

Now, let $\mathbb{R}$ be the set of real numbers. The following definition formally introduces a random variable that is bgP distributed:

Definition 2.1. *A random variable X is said to be bgP distributed with parameters* ρ, μ, $\in \mathbb{R}$ and α, β, $s > 0$, *denoted by* $X \sim bgP(\alpha, \beta, \rho, \mu, s)$, *if the probability density function (pdf) and cumulative distribution function (cdf) of X, respectively, are given by*

$$f(x)=\begin{cases}\dfrac{1}{sB(\alpha,\beta)}\left[1-\dfrac{\rho(x-u)}{s}\right]^{\frac{\beta}{\rho}-1}\left[1-\left[1-\dfrac{\rho(x-u)}{s}\right]^{\frac{1}{\rho}}\right]^{\alpha-1}, & \rho\neq 0\\ \dfrac{1}{sB(\alpha,\beta)}exp\left[-\dfrac{\beta(x-u)}{s}\right]\left[1-\exp\left[-\dfrac{\beta(x-u)}{s}\right]\right]^{\alpha-1}, & \rho=0\end{cases} \tag{2.1}$$

and

$$F(x)=\begin{cases}\dfrac{B_1\left[1-\left[1-\dfrac{\rho(x-u)}{s}\right]^{\frac{1}{\rho}};\alpha,\beta\right]}{B(\alpha,\beta)}, & \rho\neq 0.\\ \dfrac{B_1\left[1-\exp\left(-\dfrac{x}{s}\right);\alpha,\beta\right]}{B(\alpha,\beta)}, & \rho=0\end{cases} \tag{2.2}$$

The parameters ρ, μ, and s represent the shape, location, and scale parameters. The additional parameters α and β measure the skewness and tail weight of the distribution. This study considers the case when $\mu = 0$, and $\rho > 0$. According to Mahmoudi (2011), this implies that the domain becomes $\{x : 0 < x < s/\rho\}$. This is actually the effect of the location and shape parameters as described in the paper. And so, when referred $X \sim bgP(\alpha, \beta, \rho, s)$, the pdf and cdf, respectively, are

$$f(x) = \frac{1}{sB(\alpha,\beta)}\left[1-\frac{\rho x}{s}\right]^{\frac{\beta}{\rho}-1}\left[1-\left[1-\frac{\rho x}{s}\right]^{\frac{1}{\rho}}\right]^{\alpha-1}, \tag{2.3}$$

and

$$F(x) = \frac{B_1\left[1-\left[1-\frac{\rho x}{s}\right]^{\frac{1}{\rho}};\alpha,\beta\right]}{B(\alpha,\beta)}. \tag{2.4}$$

The following are the properties when $X \sim bgP(\alpha, \beta, \rho, s)$:

Theorem 2.1. *Let $X \sim bgP(\alpha, \beta, \rho, s)$ with pdf given by (2.3). Then the following hold:*

The kth raw moment of X is given by

$$\mathbb{E}[X^k] = \frac{1}{B(\alpha,\beta)}\left(\frac{s}{\rho}\right)^k \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}B[\alpha,\rho(k-j)+\beta] \tag{2.5}$$

and

The quantile function for X at level p, where $0 < p < 1$, is given by

$$Q(p) = \frac{s}{\rho}\left\{1-\left[1-\overline{B}^{-1}(p;\alpha,\beta)\right]\right\}, \tag{2.6}$$

where $\overline{B}^{-1}(p;\alpha,\beta)$ is the inverse of the cumulative distribution of a beta distribution with parameters α and β, evaluated at p.

2.2. Notations on risk measure

As described by Klugman et al. (2019), a risk measure assigns a value in $\mathbb{R}$ to a random variable describing a particular risk. For instance, VaR is a risk measure specifying an amount R in which the probability of a random variable X exceeding

R is p, for some reasonable small enough prespecified level of significance p, where $0 < p < 1$. That is,

$$\text{VaR}_X(p) := R + \inf\{r: Pr(X \geq r) = p\} \tag{2.7}$$

But because VaR fails to satisfy the subadditivity property of a coherent risk measure, one may refer to a coherent risk measure which is the CTE, also known as Tail VaR.

CTE is defined as the average of the random variable X given that Xexceeds $\text{VaR}_X(p)$ That is,

$$\text{CTE}_X(p) := \mathbb{E}[X : X > \text{VaR}_X(p)] \tag{2.8}$$

As noticed, CTE considers the average over the right-tail distribution of X (those exceeding $\text{VaR}_X(p)$) compared to the usual expectation $\mathbb{E}[X]$. The importance of dealing with the right-tail is evident when the distribution of X is known to be right heavy-tailed.

Furman and Landsman (2006) argued that CTE is not sufficient because it only considers the average over the tail but not its variability. With this, they studied the CTV which captures the deviation of X from $\text{CTE}_X(p)$ given that it exceeds $\text{VaR}_X(p)$. That is,

$$\text{CTE}_X(p): = \mathbb{E}[X - \text{CTE}_X(p)]^2 : X > \text{VaR}_X(p)] \tag{2.9}$$

To this end, the following are some remarks regarding $\text{CTE}_X(p)$ and $\text{CTV}_X(p)$:

Remarks:

If the density f of some random variable is continuous, then

$$CTE_X(p) = \frac{\int_{VaR(p)}^{+\infty} xf(x)dx}{p} \tag{2.10}$$

Expanding $\text{CTV}_X(p)$ in (2.9) yields the following formula:

$$\text{CTV}_X(p) = \mathbb{E}[X^2 : X > \text{VaR}_X(p)] - [\text{CTE}_X(p)]^2 \tag{2.11}$$

3. Results and Discussion

This section presents the closed-form formulas of the CTE and CTV of $X \sim bgP(\alpha, \beta, \rho, s)$. The first subsection is devoted to deriving the formula of the CTE followed by CTV. Application of the preceding results is presented in the next subsection. Daily closing price indices of shares of stocks in SM Prime Holdings (SMPH), Ayala Prime Holdings (APH), and Jollibee Foods Corporation (JFC) from January 01 to December 31, 2020 are used to generate possible scenarios of daily up and down of price movements. The data were retrieved in the website of Philippine Stocks Exchange, Inc. (PSEI 2020). The subsection shows how parameter estimation is done. The model is compared with distribution models such as generalized beta and generalized Pareto which are classically and commonly used in financial data modelling. This is followed by the comparison of VaR and CTE among *bgP*, generalized beta, and generalized Pareto. The models for generalized beta and generalized Pareto and their parameters are based on models presented in Klugman et al. (2019). Finally, the computation of CTE and CTV by historical simulation are compared with the values when bgP distribution is used.

As needed by the formulas of CTE and CTV, if $X \sim bgP(\alpha, \beta, \rho, s)$, then setting a level of significance using the quantile function (2.6), is given by

$$VaR_X(p) = \frac{s}{\rho}\left\{1-\left[1-\overline{B}^{-1}(1-p;\alpha,\beta)\right]\right\} \tag{3.1}$$

3.1. Conditional tail expectation and variance

Let $X \sim bgP(\alpha, \beta, \rho, s)$. Substituting (2.3) to (2.10), we get

$$CTE_X(p)\frac{1}{p}\int_{VaR_X(p)}^{s/\rho} x\frac{1}{sB(\alpha,\beta)}\left(1-\frac{\rho x}{s}\right)^{\frac{\beta}{\rho}-1}\left[1-\left(1-\frac{\rho x}{s}\right)^{\frac{1}{\rho}}\right]^{\alpha-1} dx \tag{3.2}$$

Let $u = 1-\left(1-\frac{\rho x}{s}\right)^{\frac{1}{\rho}}$. Then (3.2) becomes

$$\begin{aligned} CTE_X(p) &= \frac{s}{p\rho B(\alpha,\beta)}\int_{\hat{u}}^{1}[1-(1-u)^{\rho}](1-u)^{\beta-1}u^{\alpha-1}du \\ &= \frac{s}{p\rho}\left\{\frac{1}{B(\alpha,\beta)}\int_{\hat{u}}^{1}(1-u)^{\beta-1}u^{\alpha-1}du\right. \\ &\quad \left. -\frac{B(\alpha,\beta+\rho)}{B(\alpha,\beta)}\frac{1}{B(\alpha,\beta+\rho)}\int_{\hat{u}}^{1}(1-u)^{\beta+\rho-1}u^{\alpha-1}du\right\} \\ &= \frac{s}{p\rho}\left\{\left[1-\overline{B}(\hat{u};\alpha,\beta)\right]-\frac{B(\alpha,\beta+\rho)}{B(\alpha,\beta)}\left[1-\overline{B}(\hat{u};\alpha,\beta+\rho)\right]\right\} \end{aligned} \tag{3.3}$$

where $\hat{u}=1-\left(1-\frac{\rho Var_X(p)}{s}\right)^{\frac{1}{\rho}}$. Equation (3.3) serves as the closed-form formula for the conditional tail expectation of a *bgP*(α, β, ρ, *s*) random variable.

Now, observe that

$$\mathbb{E}[X^2:X>VaR_X(p)]=\frac{1}{p}\int_{VaR_X(p)}^{s/\rho}x^2\frac{1}{sB(\alpha,\beta)}\left(1-\frac{\rho x}{s}\right)^{\frac{\beta}{\rho}-1}\left[1-\left(1-\frac{\rho x}{s}\right)^{\frac{1}{\rho}}\right]^{\alpha-1}dx \tag{3.4}$$

Again, by letting $u=1-\left(1-\frac{\rho x}{s}\right)^{\frac{1}{\rho}}$ and setting $\hat{u}=1-\left(1-\frac{\rho VaR_X(p)}{s}\right)^{\frac{1}{\rho}}$, then the right-hand side of (3.4) yields

$$\frac{1}{p}\int_{VaR_X(p)}^{s/\rho}x^2\frac{1}{sB(\alpha,\beta)}\left(1-\frac{\rho x}{s}\right)^{\frac{\beta}{\rho}-1}\left[1-\left(1-\frac{\rho x}{s}\right)^{\frac{1}{\rho}}\right]^{\alpha-1}dx$$

$$=\frac{s^2}{p\rho^2B(\alpha,\beta)}\int_{\hat{u}}^{1}[1-(1-u)^{\rho}]^2(1-u)^{\beta-1}u^{\alpha-1}du$$

$$=\frac{s^2}{p\rho^2}\left\{\frac{1}{B(\alpha,\beta)}\int_{\hat{u}}^{1}(1-u)^{\beta-1}u^{\alpha-1}du\right.$$

$$-2\frac{B(\alpha,\beta+\rho)}{B(\alpha,\beta)}\frac{1}{B(\alpha,\beta+\rho)}\int_{\hat{u}}^{1}(1-u)^{\beta+\rho-1}u^{\alpha-1}du$$

$$\left.+\frac{B(\alpha,\beta+2\rho)}{B(\alpha,\beta)}\frac{1}{B(\alpha,\beta+2\rho)}\int_{\hat{u}}^{1}(1-u)^{\beta+2\rho-1}u^{\alpha-1}du\right.$$

$$=\frac{s^2}{p\rho^2}\left\{\left[1-\overline{B}(\hat{u};\alpha,\beta)\right]-2\frac{B(\alpha,\beta+\rho)}{B(\alpha,\beta)}\left[1-\overline{B}(\hat{u};\alpha,\beta+\rho)\right]\right.$$

$$\left.+\frac{B(\alpha,\beta+2\rho)}{B(\alpha,\beta)}\left[1-B(u;\beta+2\rho)\right]\right\}$$

Equation (3.5) completes the formula for the conditional tail variance of a *bgP*(*α, β, ρ, s*) random variable.

3.2. Data description

A total of 241 historical quotes of the daily closing price indices of SM Prime Holdings (SMPH), Ayala Prime Holdings (APH), and Jollibee Foods Corporation (JFC) from January to December 2020 were considered. These 241 data points were used to generate 240 scenarios of possible price movement, say, from today to tomorrow, relative to a unit. As an example, the closing price index of SMPH last December 28, 2020, is 37.4, and 38.6 last December 29, 2020. This generates one possible of price movement scenario with a rate of return of $i = \frac{38.6 - 37.4}{37.4} = 0.0321$. There is also a possibility of price movement with negative rate of return. Because bgP distribution only takes positive values, the paper instead considered the actual price movement relative to a unit which is 1 + i. Because 241 daily historical quotes are available: the first day for which a data point available is labeled as Day 0 price, second day as Day 1 price, and so on until the 241st day labeled as Day 240, price, then, according to what has been described above, the data produces 240 price movement scenarios, where the price movement scenario is

$$1 + \frac{\text{Day}(k+1)\text{ price} - \text{Day } k \text{ price}}{\text{Day } k \text{ price}}.$$

This generates comparable price movements of SMPH, APH, and JFC because the price movements are relative to whether 1 unit will increase or decrease by a certain amount, and not relative to the actual price. If actual prices are considered, there are large deviations among the prices of SMPH, APH, and JFC, and it is difficult to make a comparison among them. The first 20 possible scenarios are given in Table 1.

Table 1. Possible Price Movement Scenarios of SMPH, APH, and JFC (First 20 out of 240)

Scenario	SMPH	APH	JFC	Scenario	SMPH	APH	JFC
1	1.013	1.0078	1.0085	11	0.9767	0.934	0.9963
2	0.986	1.0116	0.9916	12	1.0063	0.9533	0.984
3	0.9953	1.0191	1.0188	13	1.005	1.0028	0.9981
4	0.9869	0.9875	0.9687	14	1.0124	1.0181	1.0124
5	1.0133	1.0057	0.9848	15	1.0025	1.0137	1.0255
6	1	1.0006	0.9981	16	1.0049	0.9973	0.9825
7	0.9774	0.9994	1.0261	17	0.9695	0.9858	0.9615
8	0.9915	0.9943	0.9717	18	1.0063	0.9856	0.9932
9	0.9926	1.0057	0.9757	19	1	1.0056	0.9833
10	1.0074	1.0107	1.0627	20	0.9725	1.0076	0.955

3.3 Parameter estimation

This subsection presents first the performance of bgP distribution compared to general distributions such as the generalized beta and generalized Pareto models. The performance study compares the three distribution models using the three data sets of SMPH, APH, and JFC, each having 240 data points, using maximum likelihood estimation (MLE). The comparison is based on which among the distribution models fits the data sets adequately according to the following criteria: log-likelihood value, *Akaike Information Criterion* (AIC), and *Bayesian Information Criterion* (BIC). The generalized beta and generalized Pareto models were used to generate the bgP distribution, and the comparison among the three is to determine whether bgP distribution is adequate to model fitting in comparison with its generating models (i.e., beta and Pareto, where the generalized versions are used) and according to the three data sets available.

As mentioned, to estimate the parameters of the three distribution models, maximum likelihood estimation (MLE) is used. This method requires the log-likelihood function for the parameters of a random variable , given data points. When $X \sim bgP(\alpha, \beta, \rho, s)$, Mahmoudi (2011) noted that the log-likelihood function L_1 is given by

$$L_1(\alpha,\beta,\rho,s) = (\beta-\rho)\sum_{i=1}^{n}\ln zi + (\alpha-1)\sum_{i=1}^{n}\ln(1-zi) - n\ln s - n\ln B(\alpha,\beta), \qquad (3.5)$$

where $z_i = \left(1-\frac{\rho x_i}{s}\right)^{\frac{1}{\rho}}$. Moreover, following the density functions of generalized beta and generalized Pareto in Klugman et al. (2019), the loglikelihood functions are, respectively,

$$L_3(\alpha,\theta,\tau) = n\ln\Gamma(\alpha+\tau) - n\ln\Gamma(\alpha) - n\ln\Gamma(\tau) + n\alpha\ln\theta + (\tau-1)\sum_{i=1}^{n}\ln x_i$$
$$-\sum_{i=1}^{n}\ln x_i - (\alpha+\tau)\sum_{i=1}^{n}\ln y_i, \qquad (3.6)$$

and

$$L_2(\alpha,\theta,\gamma,\tau) = n\ln\Gamma(\alpha+\tau) - n\ln\Gamma(\alpha) - n\ln\Gamma(\tau) + n\ln\gamma + \gamma\tau\sum_{i=1}^{n}\ln\frac{x_i}{\theta}$$
$$-(\alpha+\tau)\sum_{i=1}^{n}(x_i+\theta),$$

where $y_i = 1+\left(\frac{x_i}{\theta}\right)^{\gamma}$.

R software with *maxLik* package was used to numerically solve the estimates of the parameters given a data set (Henningsen and Toomet 2010). The numerical method employed is *Simulated Annealing.* Simulated annealing method does not use values of gradient and Hessian, but the log-likelihood function values only. This, in turn, is not costly in terms of the number of computations required and is not prone to accumulation of errors.

Now, summarized in Table 2 are the parameter estimates, log-likelihood values, AIC, and BIC after implementing MLE in R.

Table 2. Summary of Model Selection Criteria under SMPH, APH, and JFC Data Sets

Data	Distribution	Parameter Estimates	Log-likelihood	AIC	BIC
SMPH	bgP	$\alpha = 10.8154$ $\beta = 3.5961$ $\rho = 1.4386$ $s = 1.6553$	316.7297	-625.4593	-611.537
	Generalized Beta	$\alpha = 0.62287$ $\theta = 0.86434$ $\gamma = 22.82471$ $\tau = 7.38277$	299.7006	-591.4012	-577.479
	Generalized Pareto	$\alpha = 41.39271$ $\theta = 1.16684$ $\tau = 35.66339$	131.1453	-256.2906	-245.849
APH	bgP	$\alpha = 13.2891$ $\beta = 5.4293$ $\rho = 1.6063$ $s = 1.8465$	342.2455	-676.4909	-662.568
	Generalized Beta	$\alpha = 0.68602$ $\theta = 0.89457$ $\gamma = 22.32721$ $\tau = 3.60564$	277.6259	-547.2518	-533.329
	Generalized Pareto	$\alpha = 33.43844$ $\theta = 0.83673$ $\tau = 40.22287$	124.7108	-243.4216	-232.98
JFC	bgP	$\alpha = 11.6057$ $\beta = 6.7966$ $\rho = 2.0931$ $s = 2.3617$	350.5477	-693.0954	-679.173
	Generalized Beta	$\alpha = 0.62239$ $\theta = 0.89804$ $\gamma = 23.06133$ $\tau = 2.99245$	273.1931	-538.3862	-524.464
	Generalized Pareto	$\alpha = 39.43241$ $\theta = 1.04334$ $\tau = 37.92050$	131.0803	-256.1607	-245.719

It can be observed that among the models and using the three data sets, bgP distribution has the highest log-likelihood value, and has the lowest AIC and BIC. This favors bgP distribution as suitable one to model the data sets on SMPH, APH, and JFC compared with its underlying distributions models which is the generalized beta and generalized Pareto distributions.

The estimated parameters of $bgP(\alpha, \beta, \rho, s)$ with the corresponding p-values are summarized in Table 3. As observed, the p-values strongly suggest that the estimates are statistically significant. It is also worthy to note the estimates for ρ and s. According to how bgP model was derived, it was noted that each data point should not go beyond the upper limit s/ρ. The data sets for SMPH, APH, and JFC, respectively, have upper bounds equal to 1.1481, 1.14773, and 1.125, all of which are less than or equal to their respective upper limits s/ρ which are 1.1506, 1.1495, and 1.1283. Therefore, no problem will arise when using the estimated parameters to calculate VaR, CTE, and CTV in describing the tail behavior of the given data sets.

Table 3. Parameter Estimates with Corresponding -Values

Data	Parameter Estimates	Standard Error	-Value
SMPH	$\alpha = 10.8154$ $\beta = 3.5961$ $\rho = 1.4386$ $s = 1.6553$	0.271 0.929 0.259 0.290	2×10^{-16} 0.000108 2.78×10^{-8} 3.04×10^{-8}
APH	$\alpha = 13.2891$ $\beta = 5.4293$ $\rho = 1.6063$ $s = 1.8465$	0.738 0.692 0.169 0.194	2×10^{-16} 4.23×10^{-15} 2×10^{-16} 2×10^{-16}
JFC	$\alpha = 11.6057$ $\beta = 6.7966$ $\rho = 2.0931$ $s = 2.3617$	0.729 0.200 0.385 0.437	2×10^{-16} 0.00201 5.6×10^{-8} 6.52×10^{-8}

Now, to illustrate VaR, CTE and CTV, we consider the parameter estimates given in Table 3. By setting $p = 0.05, 0.025, 0.01$, the result of the values of the risk measures are calculated and summarized in Table 4. The value of $1-p$ represents the probability of the price movements exceeding the threshold, which is VaR. Note that CTE considers the average overall values of the random variable under those values that exceed the set threshold, which is its VaR, and from the resulting values in Table 3, the CTEs are larger than the VaRs. Moreover, CTVs, which measures the average of the squared deviation of the random variable from its CTE given that it exceeds the VaR threshold, show that over the right-tail the data depicts small variability. In turn, as dictated by the CTV, the data points tend to be very close to the CTE. Also, there is an underestimation when it comes to using VaR compared to CTE.

Table 4. Values of the Risk Measures

Data Set	Risk Measure	Values		
		At p = 0.05	At p = 0.025	At p = 0.01
SMPH	VaR	1.1146	1.1250	1.1338
	CTE	1.1263	1.1330	1.1390
	CTV	6.63×10^{-5}	3.30×10^{-5}	1.38×10^{-5}
APH	VaR	101032	1.1143	1.1245
	CTE	1.1161	1.1238	1.1311
	CTV	8.91×10^{-5}	4.95×10^{-5}	3.99×10^{-5}
JFC	VaR	1.0906	1.1003	1.1092
	CTE	1.1018	1.1083	1.1145
	CTV	6.40×10^{-5}	3.39×10^{-5}	1.52×10^{-5}

The formula for the total expectation $\mathbb{E}[X]$ and variance $V[X]$ of $bgP(\alpha, \beta, \rho, \mu, s)$ can be derived using (2.5). Doing so yields the following:

$$\mathbb{E}[X] = \frac{1}{B(\alpha,\beta)}\left(\frac{s}{\rho}\right)[-B(\alpha,\rho+\beta)+B(\alpha,\beta)] \quad (3.6)$$

and

$$V[X] = \frac{1}{B(\alpha,\beta)}\left(\frac{s}{\rho}\right)^2 [B(\alpha,2\rho+\beta)-2B(\alpha,\rho+\beta)+B(\alpha,\beta)]-(\mathbb{E}[X])^2. \quad (3.7)$$

By substituting the parameter estimates in Table 2 to (3.6) and (3.7), we get the values of the total expectation and variances summarized in Table 5.

Table 5. Values of the Total Expectation and Variance

Data Set	Measure	Values
SMPH	$\mathbb{E}$[X]	0.9849
	V[X]	0.0106
APH	$\mathbb{E}$[X]	0.9826
	V[X]	0.0086
JFC	$\mathbb{E}$[X]	0.9739
	V[X]	0.0086

It can be observed that $\mathbb{E}[X]$s are smaller than the CTEs. Small deviations from the mean are also observed but, comparatively, the total variances are larger than the variances over the right tail, which are the CTVs. Consequently, more data points concentrate over the right tail that tend to be close to CTE, as evident from the very small values of CTV so that it is expected that the CTVs are less than the total variances. Furthermore, as the CTVs are smaller than the total variances, it means that more data points are concentrated over the CTE than over the total expectation and this is expected due to long tail feature of stock return indices as it is a financial data.

To have a mere comparison about the calculated values of VaR, CTE, and CTV among other models, Tables 6 and 7 summarized the values of these risk measures when generalized beta and generalized Pareto distributions are used. These distributions are commonly used to model financial data (Jockovic 2012, He et al. 2021, and Maddala and Rao 1996). The unavailability of closed-form formulas for the VaR, CTE, and CTV under generalized beta and generalized Pareto models led to the estimation of those values by generating 100,000 random seeds from those distribution models using the estimated parameters presented in Table 2, and using these seeds to calculate VaR, CTE, and CTV. Under generalized beta distribution, most of the values of VaR are less than the VaRs under bgP. But CTEs are mostly greater than the CTEs under bgP. When generalized Pareto distribution is used, it can be observed that mostly the values of VaR and CTE are greater than the VaRs and CTEs under bgP.

Table 6. Values of the Risk Measures under Generalized Beta

Data Set	Risk Measure	Values		
		At p = 0.05	At p = 0.025	At p = 0.01
SMPH	VaR	1.1059	1.1236	1.1370
	CTE	1.1248	1.1350	1.1425
	CTV	1.44×10^{-4}	4.99×10^{-5}	1.08×10^{-5}
APH	VaR	1.1028	1.1214	1.1360
	CTE	1.1226	1.1337	1.1415
	CTV	1.66×10^{-4}	5.62 x 10-5	1.13×10^{-5}
JFC	VaR	1.090	1.1057	1.1167
	CTE	1.1062	1.1149	1.1207
	CTV	1.02×10^{-4}	3.08×10^{-5}	5.7×10^{-6}

Table 7. Values of the Risk Measures under Generalized Pareto

Data Set	Risk Measure	Values		
		At $p = 0.05$	At $p = 0.025$	At $p = 0.01$
SMPH	VaR	1.1211	1.1344	1.1424
	CTE	1.1344	1.1410	1.1452
	CTV	5.97×10^{-5}	1.64×10^{-5}	2.76×10^{-6}
APH	VaR	1.1208	1.1334	1.1419
	CTE	1.1339	1.1405	1.1445
	CTV	5.91×10^{-5}	1.55×10^{-5}	2.60×10^{-6}
JFC	VaR	1.1004	1.1129	1.1202
	CTE	1.1129	1.1189	1.1224
	CTV	4.96×10^{-5}	1.23×10^{-5}	1.83×10^{-6}

As Hull (2017) presented, the classical way of estimating the risk measures, by collecting the upper of the data points, and using these to compute the values of VaR, CTE, and CTV, is known as estimating by historical simulation. Furthermore, using this method, the computed values of VaR, CTE, and CTV are presented in Table 8. The values generated are relatively close to the computed values of VaR, CTE, and CTV under bgP model.

Table 8. Values of the Risk Measures According to Historical Simulation

Data Set	Risk Measure	Values		
		At $p = 0.05$	At $p = 0.025$	At $p = 0.01$
SMPH	VaR	1.0421	1.0556	1.0705
	CTE	1.0648	1.0813	1.1004
	CTV	7.57×10^{-4}	9.52×10^{-4}	1.16×10^{-3}
APH	VaR	1.0465	1.0643	1.0774
	CTE	1.0716	1.0863	1.1060
	CTV	6.46×10^{-4}	8.7×10^{-4}	8.58×10^{-4}
JFC	VaR	1.0517	1.0630	1.0903
	CTE	1.0751	1.0935	1.1053
	CTV	4.74×10^{-4}	2.54×10^{-4}	1.98×10^{-4}

But due to the sufficient evidence above that bgP performs well to model the three data sets as compared with its generating distributions such as the beta and Pareto, it is advantageous to calculate the risk measures using the derived formulas for those measures under bgP model for ease of calculation. Moreover, the formulas are extendable not only with stock return data sets but those can also be used in other financial and actuarial data for as long as bgP is adequate to model the data.

4. Conclusion

This paper considered the class of generalized Pareto distribution with beta distribution generator. Formulas for the raw moments and quantile function were provided under bgP model which are helpful in determining some important properties (i.e., expectation, variance, skewness, kurtosis.) An important application of the quantile function is the calculation of VaR which was also introduced in the paper. Consequently, the formulas for CTE and CTV of a random variable which is bgP distributed were provided. Application of the CTE and CTV to a real-time financial data and further comparison with the VaR, CTE, and CTV computed using the classical technique of historical simulation concluded the paper.

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