A New Compound Probability Model Applicable to Count Data

Showkat Ahmad Dar*
Department of Statistics, University of Kashmir, Srinagar (J&K), India

Anwar Hassan
Department of Statistics, University of Kashmir, Srinagar (J&K), India

Peer Bilal Ahmad
Department of Mathematical Sciences, Islamic University of Science & Technology, Awantipora, Pulwama (J&K), India

Bilal Ahmad Para
Department of Mathematical Sciences, Islamic University of Science & Technology, Awantipora, Pulwama (J&K), India

In this paper, we obtained a new model for count data by compounding of Poisson distribution with two parameter Pranav distribution. Important mathematical and statistical properties of the distribution have been derived and discussed. Then, parameter estimation is discussed using maximum likelihood method of estimation. Finally, real data set is analyzed to investigate the suitability of the proposed distribution in modeling count data.

Keywords: *Poisson distribution, two parameter Pranav distribution, compound distribution, count data, simulation study, maximum likelihood estimation.*

1. Introduction

There has been a growing concern from the last few decades to obtain flexible parametric probability distributions that can be used to model different types of data sets which cannot be quartered by classical distributions. To obtain such flexible distributions, compounding of probability distribution is comprehensive and advanced technique as it provides a very powerful way to enlarge common

**Corresponding Author: darshowkat2429@gmail.com*

parametric families of distribution to fit data sets that is not adequately fitted by classical probability distributions. Bhati et al. (2015) derived a new generalized Poisson Lindley distribution that finds applications in automobile insurance and epileptic seizure counts. Shaban (1981) built a new compound probability model for analysing count data by compounding Poisson distribution with Inverse Gaussian distribution that finds application in accidents analysis. Hassan S. Bakouch (2018) derived a count data probability model by compounding weighted negative binomial and Lindley distribution. Simon (1955) constructed a new probability model for count data by compounding Poisson with beta distribution. Pielou (1962) obtained a new compound distribution by mixing Poisson with exponential beta distribution. Sankaran (1969) constructed a class of compound Poisson distribution. Rai (1971) presented a compound of Poisson power function distribution. Mahmoudi et al. (2018) introduces a new probability model for count data by compounding Poisson with beta exponential distribution and taking Poisson distribution as parent distribution. Stacy (1962) derived a three parameter life time generalized gamma distribution. Shanker and Fesshaye (2015) introduced a new compounding probability model for count data, by compounding Poisson distribution with Lindley distribution and find its applications in biological science. Aryuyen and Bodhisuwan (2013) obtained a new compound probability model by combining Negative Binomial distribution with generalized exponential distribution. Willmot (1987) introduced the Poisson-inverse Gaussian distribution as an alternative to the negative binomial through compounding machansim. Hassan, Dar and Ahmad (2019) introduced a new compounding probability model for count data, by compounding Poisson distribution with Ishita distribution and find its applications in epileptic seizure. Lord and Geedipall (2011) showed that Poisson distribution tends to under estimate the number of zeros given the mean of the data while the negative Binomial distribution over estimates zero, but under estimate observations with a count. Umeh and Ibenegbu (2019) introduced a two parameter pranav distribution for lifetime data modeling.

In this paper we propose a new count data model which has been built by compounding Poisson distribution with two parameter Pranav distribution and taking Poisson distribution as a parent distribution, as there is a need to find more flexible models for analyzing count data.

2. Definition of Proposed Model (Poisson two parameter Pranav distribution)

If $Z|v \sim P(v)$, where v being itself a random variable following Poisson two parameter Pranav distribution with parameters ζ and η, then determining the distribution that results from marginalizing over v will be known as compound Poison distribution with that of two parameter Pranav distribution, which is denoted by PTPPD (Z; ζ, η). Our proposed model will be discrete as parent distribution is a discrete.

Theorem 1. The probability mass function of a Poisson two parameter Pranav Distribution, i.e., PTPPD (Z; ζ, η) is given by

$$P(Z=z)=\frac{\zeta^4}{(\zeta^4\eta+6)}\left[\frac{\zeta\eta(1+\zeta)^3+(z+3)(z+2)(z+1)}{(1+\zeta)^{z+4}}\right] \text{; z = 0,1,2,3,...,;}\zeta,\ \eta>0$$

Proof: The pmf of a Poisson two parameter Pranav distribution can be obtained as

$$j(z\,|\,\nu)=\frac{e^{-\nu}\nu^x}{(z)!}\ ;\ z=0,1,2,3,...,;\nu>0$$

When its parameter v follows TPPD with probability density function

$$h(\nu;\zeta)=\frac{\zeta^4(\eta\zeta+\nu^3)e^{-\zeta\nu}}{\eta\zeta^4+6};\nu>0,\zeta,\eta>0$$

The compound of Poisson distribution and two parameter Pranav distribution is given as

$$P(Z=z)=\int_0^\infty g(z\,|\,\nu).h(\nu;\zeta)d\nu$$

$$P(Z=z)=\frac{\zeta^4}{(\zeta^4\eta+6)}\left[\frac{\zeta\eta(1+\zeta)^3+(z+3)(z+2)(z+1)}{(1+\zeta)^{z+4}}\right]$$

$$;z=0,1,2,3,...,;\zeta,\eta>0$$

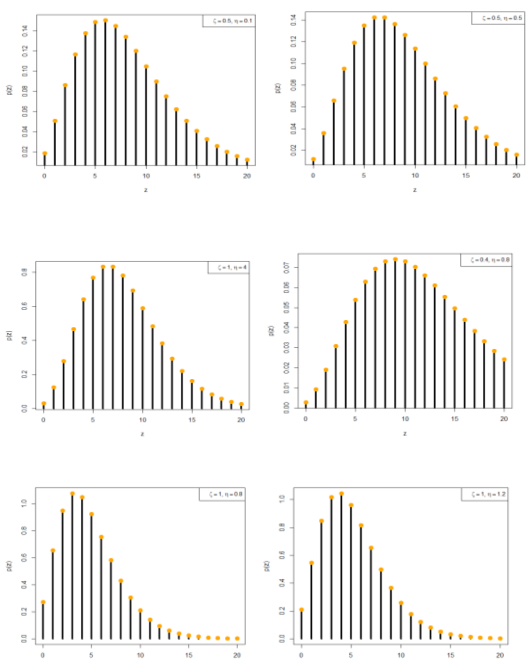

Figure 1 shows the pmf plot for the different values of η and ζ.

The corresponding cdf of Poisson two parameter Pranav distribution is given as

$$F_X(x) = 1 - \left(\frac{6+24\zeta+6z\zeta+36\zeta^2+21z\zeta^2+3z^2\zeta^2+24\zeta^2+26\zeta^2 z+9z^2\zeta^3+z^3\zeta^3+\eta\zeta^4+3\eta\zeta^5+3\eta\zeta^6+\eta\zeta^7}{(6+\eta\zeta^4)(1+\zeta)^{z+4}} \right)$$

2.1. Random data deneration from Poisson weighted Pranav distribution

In order to simulate the data from PTPPD, we employ the discrete version of inverse cdf method. Simulating a sequence of a random numbers $x_1, x_2, x_3, \ldots, x_n$ from PTPP random variable K with pmf $p(K = x_i) = p_i, \sum_{i=0}^{z} p_{i=1}$ and a cdf $F(K; \zeta, \eta)$, where z may be finite or infinite can be described as following steps:

Step 1: Generate a random number u from uniform distribution $U\,(0,1)$

Step 2: Generate random number x_i based on

$$if\ u \le p_0 = F\,(x_0 : \zeta, \eta)\ then\ K = x_0$$

$$f\ p_0 < u \le p_0 + p_1 = F(x_1 : \zeta, \eta)\ then\ K = x_1$$

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$$if \sum_{j=0}^{z-1} p_j < u < \sum_{j=0}^{z} p_j = F(x_z : \zeta, \eta)\ then\ K = x_z$$

In order to generate n random numbers $x_1, x_2, x_3, \ldots, x_n$ from PTPPD, repeat step 1 and 2 n times. We have employed R Studio software for running the simulation study of proposed model.

3. Special Case

If we put $\eta = 1$, then Poisson two parameter Pranav distribution reduces to Poisson Pranav Distribution with pmf given as

$$f(z;\eta) = \frac{\zeta^4}{(\zeta^4+6)} \left[\frac{\zeta(1+\zeta)^3 + (z+1)(z+2)(z+3)}{(1+\zeta)^{z+4}} \right]$$

4. Reliability Analysis

In this section, we have obtained the reliability and hazard rate function of the proposed PTPPD.

4.1. Reliability Function

$$R(z)=\frac{6+24\zeta+6z\zeta+36\zeta^2+21z\zeta^2+3z^2\zeta^2+24\zeta^2+26\zeta^2z+9z^2\zeta^3+z^3\zeta^3+\eta\zeta^4+3\eta\zeta^5+3\eta\zeta^6+\eta\zeta^7}{(6+\eta\zeta^4)(1+\zeta)^{z+4}}$$

4.2 Hazard Function

$$H.R=\frac{\zeta^4(\zeta\eta(1+\zeta)^3+(z+3)(z+2)(z+1))}{6+24\zeta+6z\zeta+36\zeta^2+21z\zeta^2+3z^2\zeta^2+24\zeta^2+26\zeta^2z+9z^2\zeta^3+z^3\zeta^3+\eta\zeta^4+3\eta\zeta^5+3\eta\zeta^6+\eta\zeta^7}$$

5. Factorial Moment of The Proposed Model

Theorem 5.1. The factorial moments of order s of the proposed model is given by

$$\mu_{(s)'}'=\left[\frac{\zeta^4 s!(\eta\zeta^4)!+(+s+3)(s+2)(s+1)}{(\zeta^4\eta+6)(\zeta^{4+s})}\right]$$

Proof: The sth factorial moment about origin of the PTPPD can be obtained as

$\mu_{(s)}' = E[E(Z^{(s)}| v)$, *where* $Z^{(s)=} Z(Z-1)(Z-2)\ldots(Z-s+1)$

$$\mu_{(s)}'=\int_0^\infty\left[\sum_{z=0}^\infty z^{(s)}\frac{e^{-v}v^z}{(z)!}\right].\frac{\zeta^4(\eta\zeta+v^3)e^{-\zeta v}}{\eta\zeta^4+6}dv$$

$$\mu_{(s)}'=\frac{\zeta^4}{\eta\zeta^4+6}\int_0^\infty\left[v^s\left(\sum_{z=s}^\infty\frac{e^{-v}\lambda^{z-s}}{(z-s)!}\right)\right](\eta\zeta+v^3)e^{-\zeta v}dv$$

Taking $u=z-s$, we get

$$\mu_{(s)}'=\frac{\zeta^4}{\eta\zeta^4+6}\int_0^\infty\left[v^r\left(\sum_{u=0}^\infty\frac{e^{-v}v^u}{u!}\right)\right](\eta\zeta+v^3)e^{-\zeta v}dv$$

$$\mu_{(s)'}'=\left[\frac{\zeta^4 s!(\eta\zeta^4+(+s+3)(s+2)(s+1))}{(\zeta^4\eta+6)(\zeta^{4+s})}\right]$$

6. Recurrence Relation Between Probabilities

If Z~PTPPD (ζ, η) then the pmf of Z is given as

$$P(Z=z)=\frac{\zeta^4}{(\zeta^4\eta+6)}\left[\frac{\zeta\eta(1+\zeta)^3+(z+3)(z+2)(z+1)}{(1+\zeta)^{z+4}}\right]$$

$$P(Z=z+1)=\frac{\zeta^4}{(\zeta^4\eta+6)}\left[\frac{\zeta\eta(1+\zeta)^3+(z+4)(z+3)(z+2)}{(1+\zeta)^{z+5}}\right]$$

$$\frac{P(Z=z+1)}{P(Z=z)}=\frac{\zeta\eta(1+\zeta)^3+(z+4)(z+3)(z+2)}{(1+\zeta)\zeta\eta(1+\zeta)^3+(z+3)(z+2)(z+1)}$$

$$P(Z=z+1)=\frac{\zeta\eta(1+\zeta)^3+(z+4)(z+3)(z+2)}{(1+\zeta)\zeta\eta(1+\zeta)^3+(z+3)(z+2)(z+1)}P(z)$$

7. Estimation of Parameters

In this section, we estimate the unknown parameter of the Poisson two parameter Pranav distribution by using method of maximum likelihood estimation.

7.1. Method of Maximum Likelihood Estimation

Method of Maximum Likelihood Estimation is a simple and the most efficient method of estimation. Let $Z_1, Z_2, Z_3,\ldots, Z_n$, be the random size of sample n drawn from PTPPD, then the likelihood function of PTPPD is given as

$$L(z\,|\,\zeta,\eta)=\frac{\zeta^{4n}}{(\eta\zeta^4+6)^n}\prod_{i=1}^{n}\left(\frac{(\zeta\eta(1+\zeta)^3+(z+1)(z+2)(z+3))}{(1+\zeta)^{z+4}}\right)$$

$$\log L=4n\log\zeta+\sum_{i=1}^{n}\log(\eta\zeta(1+\zeta)^3+(z+1)(z+2)(z+3))$$
$$-n\log(\eta\zeta^4+6)-(\sum_{i=1}^{n}z_i+4n)\log(1+\zeta)$$

$$\frac{\partial}{\partial\zeta}\log L=\frac{4n}{\zeta}+\sum_{i=1}^{n}\frac{(\eta+6\eta\zeta+12\eta\zeta^2+\eta\zeta^3)}{(\eta\zeta(1+\zeta)^3+(z+1)(z+2)(z+3)}-\frac{3n\eta\zeta^2}{(\eta\zeta^4+6)}-\frac{\sum_{i=1}^{n}z_i+4n}{(1+\zeta)}=0$$

$$\frac{\partial}{\partial\eta}\log L=\sum_{i=1}^{n}\frac{(\zeta(\zeta+1)^3)}{(\eta\zeta(1+\zeta)^3+(z+1)(z+2)(z+3)}-\frac{4\zeta^3}{(\eta\zeta^4+6)}=0$$

The above equations can be solved numerically by using R software 3.5.3 [12].

8. Monte Carlo Simulation

In order to investigate the performance of ML estimators for a finite sample size n using Monte Carlo simulation procedure. Using the inverse cdf method discussed in subsection 2.1, random data is generated from PTPPD. We took four random variable combinations as $\zeta = 2.8$, $\eta = 1.9$, $\zeta = 1.8$, $\eta = 1.2$, $\zeta = 0.5$, $\eta = 0.2$, and $\zeta = 0.2$, $\eta = 0.6$ to carry out the simulation study and the process was repeated 1000 times by going from small to large sample size n = (20, 50, 100, 200, 300 and 500). From Table 1, it is clear that the estimated variance and MSEs when sample size increases. Thus, the agreement between theory and practice improves as the sample size n increases. Hence, the maximum likelihood method performs quite well in estimating the model parameters of Poisson two parameter Pranav distribution.

Table 1. Average Bias, Variance and MSE of ML Estimates of Poisson Two Parameter Pranav Distribution for Different Sample Sizes

n	Parameters	$\zeta = 2.8, \eta = 1.9$				$\zeta = 1.8, \eta = 1.2$			
		Bias	Variance	MSE	Coverage probability	Bias	Variance	MSE	Coverage probability
20	ζ	-0.1212	0.00991	0.024599	0.779	0.065141	0.026776	0.031019	0.911
	η	0.17434	0.091641	0.122035	0.879	0.044127	0.061243	0.080714	0.924
50	ζ	-0.10213	0.006715	0.017145	0.901	-0.00913	0.019104	0.019187	0.929
	η	0.14012	0.061288	0.632513	0.916	0.047141	0.021208	0.021208	0.936
100	ζ	-0.0934	0.005614	0.014337	0.928	0.011207	0.007472	0.007472	0.938
	η	0.07131	0.041271	0.046356	0.931	0.016155	0.000984	0.000984	0.941
200	ζ	-0.0746	0.004124	0.009689	0.941	0.008281	0.000912	0.000912	0.948
	η	-0.0432	0.022131	0.023997	0.949	-0.00925	0.000471	0.000471	0.949
300	ζ	-0.0411	0.001971	0.003660	0.951	0.002914	0.000612	0.000612	0.951
	η	-0.0081	0.000824	0.000824	0.958	0.006714	0.000305	0.000305	0.958
500	ζ	-0.01721	0.000341	0.000341	0.961	0.006923	0.000169	0.000216	0.961
	η	-0.00910	0.000321	0.000321	0.970	0.001247	0.000106	0.000116	0.969
n	Parameters	$\zeta = 0.5, \eta = 0.2$				$\zeta = 0.2, \eta = 0.6$			
		Bias	Variance	MSE	Coverage probability	Bias	Variance	MSE	Coverage probability
20	ζ	0.352110	0.594472	0.718453	0.799	0.439618	1.281363	1.281363	0.891
	η	0.347808	0.393186	0.514156	0.839	0.395411	0.599706	0.756055	0.920
50	ζ	0.141019	0.310896	0.330782	0.906	0.485997	0.458776	0.458776	0.932
	η	0.092191	0.191920	0.200419	0.936	0.368474	0.239788	0.239788	0.939
100	ζ	-0.028951	0.198916	0.199754	0.941	0.246598	0.390818	0.980818	0.943
	η	0.058024	0.146899	0.150265	0.948	0.259943	0.193187	0.193187	0.948
200	ζ	-0.023804	0.108879	0.109446	0.951	0.138508	0.125871	0.145055	0.953
	η	0.003426	0.073616	0.073616	0.954	0.102548	0.094570	0.094570	0.959
300	ζ	0.042858	0.065758	0.065758	0.959	0.038300	0.068572	0.068572	0.964
	η	0.059003	0.039284	0.039284	0.962	0.030150	0.032064	0.032064	0.969
500	ζ	0.342880	0.007762	0.007762	0.968	0.323610	0.003848	0.003848	0.972
	η	0.327908	0.003491	0.003491	0.974	0.295564	0.006709	0.006709	0.979

9. Application of Poisson Two Parameter Pranav Distribution

In order to demonstrate the flexibility and applicability of the proposed distribution in modeling count data set, we have analyzed a data set representing automobile insurance polices (see Klugum et al. 2008), for illustrating the claim that PTPPD is providing better fits when compared to PLD, GD, PD, ZIPD and NBD. The data has a long right tail and approaches to zero slowly. The data sets are given in Table 2.

Table 2. Dataset Representing Automobile Insurance Polices Counts (see Klugman et al. (2008))

Z	0	1	2	3	4	5	6	7	8
Observed Counts	7840	1317	239	42	14	4	4	1	0

For estimation of parameters of the distribution, maximum likelihood method and R software has been used. Parameter estimates, standard errors and model function of the fitted distribution is given in Table 3.

Table 3. Parameter Estimates and Standard Errors for Ffitted Distributions for Dataset 2 (Estimated parameters and standard error for fitted distributions for dataset representing automobile insurance polices counts)

Distribution	Parameter Estimates (Standard Error)	Model function
PTPPD	ζ = 5.62 (0.4) η = 0.08 (0.06)	$P(Z=z)=\frac{\zeta^4}{(\zeta^4\eta+6)}\left[\frac{\zeta\eta(1+\zeta)^3+(z+3)(z+2)(z+1)}{(1+\zeta)^{z+4}}\right]$ $Z=0,1,2,3,\ldots,;\ \zeta\,\eta>0$
PD	ν = 0.21 (0.04)	$p(z)=\frac{e^{-\nu}\nu^z}{z!}\quad \nu>0;\ z=0,1,2,\ldots$
PLD	η = 5.39 (0.11)	$p(z)=\frac{\eta^2(z+\eta+2)}{(\eta+1)^{z+3}}\quad Z=0,1,2,\ldots\theta>0$
GD	p = 0.82 (0.03)	$p(z)=q^z p\quad 0<q<1;\ q=1-p;\ z=0,1,2,\ldots$
NBD	r = 0.70, p = 0.77 (0.2, 0.04)	$p(z)=\binom{z+r-1}{z}p^r q^z,\quad z=0,1,2,\ldots$ $r>0$ and $0<p<1$
ZIPD	η = 0.46, σ = 0.54 (0.02, 0.02)	$p(z)=\begin{cases}\eta+(1-\eta)\frac{e^{-\sigma}\sigma^z}{z!}, & \sigma>0;\ z=0\\(1-\eta)\frac{e^{-\sigma}\sigma^z}{z!}, & \sigma>0;\ z=0,1,2,..\end{cases}$ $0<\eta<1;\ \sigma>0$

We have fitted Poisson two parameter Pranav distribution (PTPPD), zero inflated Poisson distribution (ZIPD), geometric distribution (GD), Poisson Lindley distribution (PLD), negative binomial distribution (NBD) and Poisson distribution (PD) to the data set given in Table 2. In order to check the goodness of fit of the model and estimation of parameters of the model, Person's chi-square test R studio statistical software has been used. The results are given in Table 4. It is clear from the expected frequencies and the corresponding value of chi-square that Poisson two parameter Pranav distribution provides a satisfactorily better fit for the data set representing automobile insurance claims as compared to other competing models. It is also clear from Figure 2 the values of expected frequencies that Poisson two parameter Pranav distribution provides a closer fit than that provided by other competing models.

Table 4. Fitted PTPPD and Other Competing Models to a Dataset Representing Automobile Insurance Polices

Z	Observed Counts	PD	ZIPD	GD	PLD	NBD	PTPPD
0	7840	627.9	7840	7790.9	7757.7	7879.2	7816.3
1	1317	1703.2	1272.4	1375.25	1381.3	1268.5	1334.6
2	239	2310	296.55	242.75	241.5	248	248.1
3	42	2088.7	46	42.85	41.75	51.3	45.6
4	14	1416.5	5.4	7.55	7.15	10.9	10.1
5	4	768.5	0.5	1.35	1.2	2.4	3.1
6	4	374.4	0.1	0.25	0.2	0.5	2.6
7	1	134.6	0.1	0.1	0.1	0.1	0.5
8	0	64.3	0.1	0.1	0.1	0.1	0.01
Degrees of freedom		4	2	3	3	2	3
Chi-Statistic Value		16517	61.2	23.5	27.4	32.2	3.95
p-value		0	0	0	0	0	0.266

AIC (Akaike information criterion) and BIC (Bayesian information criterion) criterions has been used for comparing our proposed model with other competing models. The lower values of AIC and BIC corresponds to better fitting of model.

As it is clear from Table 5, that the Poisson two parameter Pranav distribution has lesser values of AIC and BIC as compared to other competing models, hence we can concluded that the Poisson two parameter Pranav distribution leads to a better fit than the other competing models for analyzing the data set given in Table 2.

Table 5. AIC, BIC and -logl for Fitted Models to a Dataset Representing Automobile Insurance Polices

Criterion	PD	ZIPD	GD	PLD	NBD	PTPPD
-logl	5359.5	5375.6	5354.7	5356.25	5358	5348.7
AIC	10725	10755.2	10755.2	10714.5	10718	10701.4
BIC	10746.4	10769.5	10769.5	10721.7	10720.2	10701.8

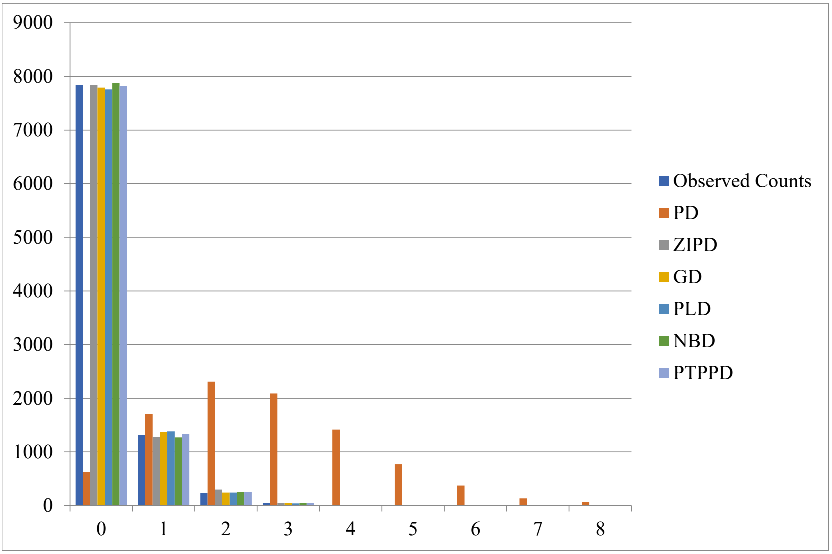

Figure 2. Graphical overview of fitted models to dataset given in Table 2

10. Conclusion

In this paper, we discussed a new model which has been built using compounding technique. Statistical and mathematical properties such as reliability, hazard rate and moments have been discussed. Finally, a real data set is discussed to demonstrate the fitness and applicability of the Poisson two parameter Pranav distribution in modeling count dataset.

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