A Class of Ratio-Cum-Product Type Exponential Estimators under Simple Random Sampling

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In this paper, a class of ratio-cum-product type exponential estimators have been proposed under simple random sampling to estimate the population mean. The proposed class of estimators has been compared with the other existing estimators. We have compared the efficiency of the proposed class of estimators with the other standard estimators and found through empirical study that the previous estimators are inferior to the present proposed class of estimators. The population used in the empirical study are all varying very much from each other and thus it demonstrate the superiority of the present estimators under all type of situation.

Keywords: auxiliary variable, study variable, simple random sampling, ratio type estimators, product type estimators, bias, MSE

1. Introduction

A number of sampling techniques rely on the possession of advanced information about an auxiliary variable (say X). The basic goal of these techniques is to obtain improved estimators of population mean $\overline{Y}$ of a study variable Y by taking the advantage of correlation between Y and X. There are plenty of situations where we are interested in estimating the population mean of the variable of interest in presence of an auxiliary variable. Under the availability of information on an auxiliary variable, several types of estimators have been developed in sampling literature because of its intuitive appeal, computational simplicity and applicability to a general design. Hence, many researchers invested their efforts toward improving the efficiency of ratio and product type estimators with the intent to get more efficient estimators of the population mean than the linear regression estimator. Some noteworthy contributions in this direction have been made by Cochran (1940), Robson (1957), Murthy (1964), Singh (1967), Sahai (1979), Bahl and Tuteja (1991), Kadilar and Cingi (2005), Singh and Vishwakarma (2008), Shabbir et al. (2014), and many others.

Let $U = (U_1, U_2,...U_N)$ be a finite population consisting of N units. Also let (x_i, y_i) this be the observations on the i^{th} unit of the variables (X, Y). Further let $(\overline{X}, \overline{Y})$ and $(\bar{x}, \bar{y})$ be the respective population means and sample means of (X, Y). Assuming that the population mean $\overline{X}$ is known, the population mean $\overline{Y}$ is estimated by selecting a sample of size n from the population U of size N with the help of simple random sampling without replacement(SRSWOR) scheme.

The classical ratio and product estimators for $\overline{Y}$ are given by

$$\bar{y}_R = \bar{y}\left(\frac{\overline{X}}{\bar{x}}\right) \tag{1}$$

$$\bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\overline{X}}\right) \tag{2}$$

Bahl and Tuteja (1991) suggested the following exponential type ratio and product estimators for $\overline{Y}$.

$$\bar{y}_{Re} = \bar{y}\, exp\left[\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right] \tag{3}$$

$$\bar{y}_{Pe} = \bar{y}\, exp\left[\frac{\bar{x}-\overline{X}}{\bar{x}+\overline{X}}\right] \tag{4}$$

2. Proposed Class of Estimators in Simple Random Sampling

Motivated by the preceding estimators, we propose the following class of estimators for the population mean $\overline{Y}$ in SRSWOR.

$$T = \bar{y}\, exp\left[\frac{\left(\overline{X}/\bar{x}\right)^{\alpha}-1}{\left(\overline{X}/\bar{x}\right)^{\alpha}+1}\right] \tag{5}$$

where α is a real number to be determined so as to minimize the mean square error (MSE) of the proposed class T.

It can be seen that for $\alpha = 1$, the proposed class reduces to $\bar{y}_{Re}$, and for $\alpha = -1$, the proposed class reduces to $\bar{y}_{pe}$.

3. Bias and Mean Squared Error of Proposed Estimator

In order to obtain the bias and MSE of T, we write

$$\bar{y} = \bar{Y}(1 + e_0), \bar{x} = \bar{X}(1 + e_1)$$

such that,

$$\left.\begin{aligned} &E(e_0) = E(e_1) = 0 \\ &E(e_0^2) = f_1 C_Y^2 , E(e_1^2) = f_1 C_X^2 , \\ &E(e_0 e_1) = f_1 \rho_{yx} C_Y C_X , \end{aligned}\right\} \tag{6}$$

where

$$f_1 = \left(\frac{1}{n} - \frac{1}{N}\right), C_Y = \frac{S_Y}{\bar{Y}}, C_X = \frac{S_X}{\bar{X}}, \rho_{yx} = \frac{S_{YX}}{S_Y S_X},$$

$$S_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2 , S_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2 ,$$

$$S_{YX} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})$$

Now expressing T in terms of e's, we have

$$T = \bar{y}(1+e_0)\, exp\left[\frac{1-(1+e_1)^{\alpha}}{1+(1+e_1)^{\alpha}}\right] \tag{7}$$

On simplifying, we get

$$T = \bar{Y}\left[1 + e_0 - \frac{\alpha}{2}(e_1 + e_0 e_1) + \frac{\alpha^2}{8} e_1^2\right] \tag{8}$$

or $$T - \bar{Y} = \bar{Y}\left[e_0 - \frac{\alpha}{2}\left(e_1 + e_0 e_1\right) + \frac{\alpha^2}{8} e_1^2\right] \tag{9}$$

Taking expectation on both side of (9) and using results in (6), we get the bias of the proposed class T as

$$Bias(T) = f_1 \frac{\alpha}{2} \bar{Y} C_X^2 \left[\frac{\alpha}{4} - \rho_{xy}\frac{C_X}{C_Y}\right] \tag{10}$$

Further, squaring both sides of (9), neglecting the terms of e's having degree greater than two, taking the expectation and using results in (6), we obtain the MSE of T to the first degree of approximation as

$$MSE(T) = f_1\overline{Y}^2\left(C_Y^2 - \alpha\rho_{yx}C_YC_X + \frac{\alpha^2}{4}C_X^2 \right) \tag{11}$$

Also, the MSEs of $\overline{y}_R$, $\overline{y}_P$, $\overline{y}_{Re}$ and $\overline{y}_{Pe}$ to the first degree of approximation, are given by

$$MSE(\overline{y}_R) = f_1\overline{Y}^2\left(C_Y^2 + C_X^2 - 2\rho_{X}\ C_YC_X\right) \tag{12}$$

$$MSE(\overline{y}_P) = f_1\overline{Y}^2\left(C_Y^2 + C_X^2 + 2\rho_{YX}C_YC_X\right) \tag{13}$$

$$MSE(\overline{y}_{Re}) = f_1\overline{Y}^2\left(C_Y^2 - \rho_{yx}C_YC_X + \frac{1}{4}C_X^2 \right) \tag{14}$$

$$MSE(\overline{y}_{Pe}) = f_1\overline{Y}^2\left(C_Y^2 + \rho_{yx}C_YC_X + \frac{1}{4}C_X^2 \right) \tag{15}$$

3.1. Optimum value of α

The value of α for which we minimizes the MSE of T , is obtained by using the condition:

$$\frac{\partial}{\partial\alpha}MSE(T) = 0$$

Thus, the optimum value of α is

$$\alpha_{opt} = 2\rho_{yx}\frac{C_Y}{C_X} \tag{16}$$

and on substituting α_{opt} in (11) , we get the minimum MSE of T as

$$MSE(T)_{min} = f_1\overline{Y}^2C_Y^2(1-\rho_{yx}^2) \tag{17}$$

Note that the optimum choice of the constant involves unknown parameters. These quantities can be guessed quite accurately experience gathered in due course of time or sampled data or conducting a pilot survey.

4. Efficiency Comparison

It is well known that the variance of sample mean $\bar{y}$ is

$$Var(\bar{y}) = f_1 \bar{y}^2 C_Y^2 \tag{18}$$

In this section, the effective ranges of α are obtained for which the MSE of T are less than the MSEs of other popular estimators. For making the efficiency comparisons of the proposed class T with other existing estimators, we have from (11),(12),(13),(14),(15) and (17)

$MSE(T) < Var(\bar{y})$ if

$$\alpha < 4\rho_{xy} \frac{C_Y}{C_X} \tag{19}$$

$MSE(T) < MSE(\bar{y}_R)$ if

$$\alpha < 4\rho_{YX} \frac{C_Y}{C_X} - 2 \tag{20}$$

$MSE(T) < MSE(\bar{y}_P)$ if

$$\alpha < 4\rho_{YX} \frac{C_Y}{C_X} + 2 \tag{21}$$

$MSE(T) < MSE(\bar{y}_{Re})$ if

$$\alpha < 4\rho_{yx} \frac{C_Y}{C_X} - 1 \tag{22}$$

$MSE(T) < MSE(\bar{y}_{Pe})$ if

$$\alpha < 4\rho_{xy} \frac{C_Y}{C_X} + 1 \tag{23}$$

In order to get more insight, we may further investigate under the assumption of bivariate normality of Y and X and equality of coefficients of variation of Y and X, and show the preferred estimators for different ranges of ρ in the table below.

Table 1. Comparison between different various Estimators of $\bar{Y}$

Rang of ρ	Preferred Estimator	Rang of ρ	Preferred Estimator
$0 < \rho < 0.25$	$\bar{y}$	$0 < \rho < -0.25$	$\bar{y}$
$0.25 < \rho < 0.75$	$\bar{y}_{Re}$	$-0.25 < \rho < -0.75$	$\bar{y}_{Pe}$
$0.75 < \rho < 1$	$\bar{y}_R$	$-0.75 < \rho < -1$	$\bar{y}_P$
$0 < \rho < 1$	T	$0 < \rho < -1$	T

From the Table 1, we can easily seen the performance of proposed class of estimators have been uniformly better than other estimators in the whole range of ρ.

5. Empirical Studies

To examine the merits of the proposed class over other existing estimators we have to consider the following population. All the below population consists of the different range of coefficient of variation and correlation coefficient and sample drawn using simple random sampling technique. Hence, it will be very much helpful in demonstrating the superiority between the proposed estimator and the other standard estimators mentioned above. The description of the parameters of populations are given below:

Population I - [Source: Murthy (1967, p. 228)]
X : fixed capital
Y : output
$N = 80$, $n = 30$, $\overline{Y} = 5182.64$; $C_y = 0.3542$; $C_x = 0.7507$; $\rho_{xy} = 0.9413$;

Population II- [Source: Murthy (1967, p.228)]
X : number of workers
Y : output
$N = 80$, $n = 30$, $\overline{Y} = 5182.64$; $C_y = 0.3542$; $C_x = 0.9484$; $\rho_{xy} = 0.9150$;

Population III [Source: Das (1998)]
X : number of agricultural laborers for 1961
Y : number of agricultural laborers for 1971
$N = 278$, $n = 70$, $\overline{Y} = 39.0680$; $C_y = 1.4451$; $C_x = 1.6198$; $\rho_{xy} = 0.7213$;

Population IV [Source: Steel and Torrie (1960, p. 282)]
X : chlorine percentage
Y : log of leaf burn in sacs
$N = 30$, $n = 12$, $\overline{Y} = 0.6860$; $C_y = 0.4803$; $C_x = 0.7493$; $\rho_{xy} = 0.4996$;

Population V [Source: Dobson (1990, p. 47)]
X : initial white blood cell count
Y : survival time leukemia patient
$N = 20$, $n = 8$, $\overline{Y} = 0.6860$; $C_y = 0.2017$; $C_x = 0.7493$; $\rho_{xy} = 0.54709$;

Here we have computed the Percentage Relative Efficiencies (PREs) of different suggested estimators of $\overline{Y}$ with respect to $\overline{y}$ using the formula:

$$PRE(\xi) = \frac{V(\bar{y})}{MSE(\xi)} \times 100 \quad (24)$$

where $\zeta = \bar{y}, \bar{y}_R, \bar{y}_P, \bar{y}_{Re}, \bar{y}_{Pe}$ and T and the findings are presented in Table 2.

Table 2. Percentage Relative Efficiencies of various Estimators of $\bar{Y}$

Estimators	Population1	Population2	Population3	Population4	Population5
$\bar{y}$	100.00	100.00	100.00	100.00	100.00
$\bar{y}_R$	66.58	30.58	156.39	*	*
$\bar{y}_P$	*	*	*	53.33	9.31
$\bar{y}_{Re}$	781.39	292.08	197.78	*	*
$\bar{y}_{Pe}$	*	*	*	119.81	41.36
T	877.54	614.35	208.45	132.63	142.71

**Data is not applicable*

6. Conclusion

Table 2 clearly indicates that the proposed class of ratio-cum-product estimator T has maximum PREs in all the five populations data sets and hence is more efficient than the other estimators i.e, the usual unbiased estimator $\bar{y}$, the ratio estimator $\bar{y}_R$, the product estimator $\bar{y}_P$, exponential type ratio estimator $\bar{y}_{Re}$, exponential type product estimator $\bar{y}_{Pe}$ and T. Hence, the proposed class should be preferred in practice.

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