Modelling Portfolio Risk and Diversification Effects of a Portfolio Using the Exponential Distribution – Bivariate Archimedean Gumbel Copula Model

Owen Jakata[1,2] and **Delson Chikobvu**[2]

[2]**Faculty of Natural and Agricultural Sciences,**
Department of Mathematical Statistics and Actuarial Science,
University of Free State, Bloemfontein 9300, South Africa

ABSTRACT

This study uses the Archimedean Gumbel copula model to construct the dependence structure and joint probability distributions using the Exponential Distribution as the marginal distribution to asset returns. The main objective of this study is to estimate the diversification effects of investing in a portfolio consisting of two financial assets, viz: the South African Industrial and Financial Indices. The Exponential Distribution is used as the marginal distribution of the returns, instead of the Normal distribution, to better characterise the financial returns of the two assets. The scatterplots indicate that the dependence in gains, as well as the losses are better captured using the Archimedean Gumbel copula. Monte Carlo simulation of an equally weighted portfolio of the two financial assets is used to model and quantify the risk of the resultant portfolio. The results confirm that there are benefits in diversification, since the riskiness of the portfolio is less than the sum of the risk of the two financial assets. It is less risky to invest in diversified portfolios that includes assets from the two different industries/stock markets. Due to dependence and contagion between Global stock markets, the findings of this study are useful information for the local and international investors seeking a portfolio which include developing countries' stock market Indices containing, say the South African financial assets. This study provides investors with a framework to quantify diversification effects, which allows for the avoidance of extreme risks, whilst benefiting from extreme gains.

Keywords: *Expected Shortfall. Monte Carlo simulation, Value-at-Risk*

I. INTRODUCTION

According to Alqahtani, et al. (2020), financial integration has been of growing interest to investors and practitioners since the Global Financial Crisis (GFC) of 2008-2009 as it reflects the increased association of Global stock markets arising from rapid globalisation. Global stock markets have become highly dependent and correlated to each other. However, developing countries' markets are less dependent on major Global stock markets. Furthermore, the high correlation/dependence between the assets reduces the benefit of diversification to a minimum, thus making the risk of the portfolio comparable to any of the individual equities (Stanga, 2008). This study estimates the portfolio risk and diversification effects/benefits of the upside and downside stock market risk of the South Africa Industrial and Financial Indices.

[1] Corresponding Author: owenjakata@rocketmail.com

One of the main challenges in risk management is the aggregation of individual risk factors. The problem becomes much more challenging when modelling fully dependent random variables or, when one does not know the joint distribution to use in order to determine the dependence structure (co-movement) between the given individual financial risk factors. A copula is generally used to model dependence structure (co-movement) between individual financial risk factors. A copula is used to construct the dependence structure/joint probability distributions used to estimate portfolio risk and diversification effects thereof. According to Oyenubi (2017), the diversification between different assets tends to contribute to portfolio performance. This implies that if there are more independent sources of variation in a portfolio, this would mean that there will be less susceptibility to shocks from specific components. Thus, the volatility will be kept as low as feasible.

According to Sukcharoen and Leatham (2018), there is higher dependence and correlation between Global stock markets (which mainly excludes developing countries' stock markets). Diversification benefits of international investing are not optimized if developing countries' stock markets are not included. Investors and practitioners can mitigate the impact of rising dependence and inflation between Global stock markets by investing in portfolios which include developing countries' stock markets, e.g., containing the South African stock market financial assets. This can benefit investors in potentially improving investment diversification, as developing countries' markets are less correlated/dependent with other Global stock markets. Efforts have been made to investigate the inter-dependence of stock returns at industrial sector level (Choudhry and Osoble, 2015). In this study two industrial sectors viz: the South African Industrial and Financial sectors are investigated. This study is confined to a bivariate case, although in principle the ideas discussed can be applied to higher dimension models.

The portfolio risk measures adopted in this study are the Value at Risk (VaR) and Expected Shortfall (ES). According to Hotta, et al. (2008), VaR is the primary risk metric used by investors and practitioners to estimate the market risk, due to its simplicity, ease of computation and applicability. According to Daníelsson, et al. (2013), VaR is not a coherent risk measure and therefore is not sub-additive. However, VaR is sub-additive in most practical situations, which is then consistent with the diversification principle of modern portfolio theory. This property of sub-additivity is discussed in Section III.6. It is a mathematical description of diversification and therefore represents the benefits of diversification for a portfolio (Stanga, 2008), to be estimated in this study.

In this study, the Exponential distribution marginals are used to estimate individual risks in the returns of the two financial assets. The two financial assets are relatively heavy-tailed, hence arguing for the use of the Exponential marginal distributions. The Exponential distribution is used since it is leptokurtic in nature, as it tends to exhibit a relatively heavier-tail and excess peak from the mean than the Normal distribution. Yeap et al. (2020) proposed the use a combination of the heavy-tailed statistical distributions and copula functions to overcome the shortcomings of the traditional VaR methodology which uses the Normal distribution. Ghorbel and Trabelsi (2009) states that, a combination of heavy-tailed distributions and copula models offer to investors and practitioners, a powerful tool to model a portfolio with extreme risk, than using a traditional Normal distribution-based univariate and multivariate methods which results in significantly less accurate risk estimates. The Exponential distribution-Archimedean Gumbel copula is therefore adopted in this study. This model is used to estimate portfolio risk and is then used to calculate diversification effects/benefits.

1. Statement of the Problem

In risk management, the main challenges of estimating portfolio risk and the calculation of diversification effects, is the aggregation of individual risk factors. The problem becomes more challenging when one does not know which copula to use in order to determine the dependence structure (co-movement) between the given individual risk's factors. The main objective of this paper is to determine the appropriate copula, estimate portfolio risk and calculate the diversification effects/benefits resulting from the uncertainties in the dependence structure (co-movement) between two risky assets. The extreme Gumbel copula model was found to be appropriate to construct the dependence structure (co-movement). The Exponential distribution is used as the marginal distribution to better characterise the financial returns of the two risky assets. The Exponential distribution as a marginal distribution is known to be sufficient to cater for varying degrees of skewness and kurtosis in the data sets, and provides for the relatively heavy-tailed of the returns from the two assets than the Normal distribution does. Investors and practitioners seek diversification to reduce the portfolio risk inherent in investing in the risky assets. This study uses the Exponential distribution-Archimedean Gumbel copula and Monte-Carlo simulation to estimate the portfolio risk in order to account for diversification effects/benefits using a portfolio of the two assets, viz: the South African Industrial and Financial Indices.

2. Justification of the Study

Estimation of the portfolio risk and ddiversification effects are two of the many metrics that an organisation should estimate before making an investment decision. Diversification is a strategy of spreading investments, and allows for a restricted amount of exposure to any one form of asset, industry, or country. The goal of this technique is to lessen the overall portfolio's volatility. Diversification does not guarantee success or protect against loss. Limiting the impact of volatility on a portfolio is the main objective of diversification. The number of stocks in a portfolio and the correlation/dependence pattern both affect diversification. A portfolio's level of diversification is increased by adding equities, which reduces risk. Literature on estimation of investment risk (portfolio risk and diversification effects/benefits) in the context of South Africa is rather limited.

3. Objectives of the Study

The main objective of this study is to determine the appropriate statistical distribution and copula, and then estimate portfolio risk of the two assets, viz: the South African Industrial and Financial Indices whilst accounting for the diversification effects after investing in an equally weighted portfolio of the two assets.

The specific objectives are to:

- Fit an appropriate statistical (Exponential) distribution to the South African Industrial Index (J520) and the South African Financial Index (J580) returns and estimate univariate risk of VaR and ES.
- Determine the appropriate copula to fit the bivariate returns data (in this study it was found to be the Archimedean Gumbel copula for both the losses and gains).
- Fit a bivariate joint distribution using the Archimedean Gumbel copula model to the gains and losses separately.

- Estimate portfolio risk (VaR and ES) using Exponential distribution-Archimedean Gumbel copula model and interpret these associated risk measures for the portfolio.
- Estimate the diversification effects thereof.

The originality of work in this paper should be viewed within the context of applying already established statistical methodology to provide new insights into the complex dependence between returns on assets, and by demonstrating the impact on the South Africa market. The approach is a departure from the traditional Normal distribution univariate risk measures and the linear correlation when assessing a portfolio risk. The EVT- Copula approach is used as a tool for understanding and predicting risk from existing data on the two returns in forming the portfolio. The current paper also aims to fill an important gap, investigating the portfolio risk on two prominent financial assets in South Africa. The contribution of this study is in extending the empirical literature to more accurate risk estimation, and calculation of diversification effects of a portfolio using the Exponential Distribution-Archimedean Gumbel copula model. The portfolio risks estimated are important as they provide statistical information which can be used as a framework to help investors and practitioners calculate the diversification effects more accurately, especially in the case of the extreme risky scenarios. The study is also a reference for similar research on the portfolio risk when there is dependence in asset returns.

II. REVIEW OF LITERATURE

Studies in literature in which the Archimedean Gumbel copula functions are used in combination with statistical parent distributions and or extreme value distributions are discussed in this section.

Emamverdi (1999), investigated the estimation of portfolio VaR using an approach combining Copula functions, EVT and GARCH models. The study investigated the dependence structure between TEPIX Index (Iran) and NASDAQ Index (United States of America). The study used the GARCH model and the EVT method to model the marginal distributions of each log returns series for the indices and then use Copula functions to estimate the dependence structure (co-movement) of the bivariate distribution. The results revealed that Copula functions estimate portfolio VaR more accurately than the traditional methods.

Mwamba and Mokwena (2013), estimated equity portfolio risk using the GPD-Archimedean copula approach. The study used eight sets of data consisting of the following stock market indices, viz: the JSE/FTSE All Share Index (South Africa), Bovespa Index (Brazil), Indice de Precios Cotizaciones Index (Mexico), Shanghai Composite Index (China), S&P500 (United States of America), FTSE100 (Britain), DAX (Germany) and CAC40 (France). The results showed that the Gumbel Copula and Clayton Copula are appropriate for modelling the upper tail dependence and lower tail dependence of the return distribution respectively.

Hotta, et al. (2008), proposed a method for estimating portfolio VaR using the GPD-Gumbel copula model. The study used a portfolio of two stock market Indices, viz: IBOVESPA and MERVAL from Brazil. The losses/ (on the left tail of return distribution) were modelled using the GPD as the marginal distribution. The Gumbel Copula was used to estimate the dependence structure/joint probability distribution of the two Indices. The model was used to estimate portfolio VaR of the two indices and compared to other traditional methods. The results revealed that the GPD-Gumbel Copula model provides more accurate portfolio risk estimates than other traditional methods.

Clemente and Romano (2005), used Gaussian Copula and Student-t-Copula with Normal distribution and the GPD marginals to estimate portfolio VaR. The researchers used the Monte Carlo method for estimating and optimising the portfolio risk of 20 Italian equities. The dependence structure was modelled using Gaussian copula and Student-t-copula with Normal distribution and GPD marginals. The aim was is to obtain an accurate estimate of the portfolio VaR. Their findings were that the GPD-copula model provided better VaR estimates than other traditional VaR models.

Inanoglu (2005), applied the GPD-copula to model a non-linear dependence structure(co-movement) of financial returns and estimated diversification effects. The GPD was used as the marginal distribution to characterise extreme returns for the lower and upper tails in order to quantify the VaR and ES. The simulated sample consisted of financial returns data from United States of America financial institutions. The results revealed that the GPD-copula provides more accurate portfolio VaR and ES estimates than historical simulation. The estimated diversification effects for the U.S financial institutions ranged from 20% to 70%.

Mendes and de Souza (2004), used copulas to model the dependence structure of a portfolio consisting of two financial indices returns, viz: IBOVESPA(Brazil) and S&P500 (USA). When compared to the Student t-copula under heavier-tails, the study found little difference in the benefits of diversification under the Gaussian copula.

There are many other studies in literature that have applied the Archimedean Copula functions with the different statistical distributions as marginals to estimate portfolio risk, but this study extends these studies by not only estimating portfolio VaR but also estimating diversification effects/benefits. There are limited studies that use a combination of the Exponential distribution marginals and Archimedean Gumbel Copula to estimate portfolio risk and even diversification effects/benefits. The study uses the full dataset in distribution modelling by using the Exponential distribution as marginal distributions for the South African Industrial Index (J520) and the South African Financial Index (J580) returns. This study differs from the other studies as it uses the unconditional approach which involves the direct application of Exponential distribution as marginals to the stock returns.

III. METHODOLOGY

This study uses the Exponential Distribution-Archimedean Gumbel Copula model to estimate portfolio risk and calculate diversification effects thereof. These two components in the model are discussed in the section below.

1. Statistical Parent Distributions

There are many different statistical parent distributions including: the Exponential distribution, the Weibull distribution, the Gamma distribution and the Burr distribution to describe a return distribution. The statistical parent distributions model the main body of the data. These distributions are also moderately heavy tailed, and the Burr distribution can be described as heavy tailed. In this study, the Exponential distribution is deemed appropriate using various goodness of fit tests. The Exponential distribution uses the full data set in the modelling process. The data is applied to the Exponential Distribution-Archimedean Copula model, to estimate portfolio risk, and diversification effects. The distribution is fitted to both the gains and losses returns separately. The losses were formed by taking negative log returns and multiplying by negative one to make them positive as the Exponential distribution does not support negative random variables. The Maximum Likelihood Estimate (MLE) method is used to estimate the parameters for the fitted distributions.

2. The Exponential Distribution

Bryson (1974), describes a heavy-tailed distribution as having a tail that is heavier than a Normal distribution. The Exponential distribution has a heavier tail than a Normal distribution. The Exponential distribution gives a good starting point relative to our presumption on the nature of the returns data. According to Chan, et al. (2016), the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the Exponential distribution are respectively denoted as:

$$f(x;\lambda) = \lambda e^{-\lambda x} \tag{1}$$

$$F(x;\lambda) = 1 - e^{-\lambda x} \tag{2}$$

where x represents the log returns and $\lambda > 0$ is the rate parameter. The MLE parameter for the Exponential distribution is given in the following theorem:

Theorem (Chan et al. 2016): If X is exponentially distributed with the pdf given in Equation (1) where $\lambda > 0$, then the MLE of λ is given as:

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}, \tag{3}$$

where *n* is the sample size for the data set.

3. Exponential Distribution Risk Measures

In this section, the formulas used to calculate and quantify risk in the South African Industrial Index (J520) and the South African Financial (J580) Index returns data, namely: VaR and the ES for the proposed parent distributions are discussed.

Exponential distribution VaR and ES equations (Chan et al. 2016). The VaR and ES are calculated using formulas in Equations (4) and (5) below.

$$VaR_p(\mathrm{X}) = -\frac{1}{\lambda} \log(1 - \mathrm{p}) \tag{4}$$

$$ES_p(\mathrm{X}) = -\frac{1}{p\lambda}\{\log(1-p)\, p - p - \log(1-p)\} \tag{5}$$

for $x > 0$, $0 < p < 1$, and $\lambda > 0$, the scale parameter.

4. Modelling Tail Dependence

The concept of tail dependence deals with the joint probability of extreme events that can occur in the upper tail or lower tail, of say a bivariate distribution. In this study the Archimedean Gumbel copula is used in the analysis of the bivariate portfolio, viz: the South African Industrial Index (J520) and the South African Financial (J580) Index returns.

In many financial applications, there is strong lower tail dependence between extreme maximum losses than between extreme maximum gains (Embrechts, et al. 2001). Such asymmetries can be easily modelled using Archimedean copulas since they are flexible in allowing various forms of dependence. The Archimedean copulas are somewhat simple as they have closed form expressions. The copula-based approach can be applied and fitted to marginal distributions of several variables (Salleh, et al. 2016). Gumbel (1960) stated that the Archimedean Gumbel Copula is an extreme value copula which is asymmetric, and exhibits greater upper tail dependence in one corner and is applied in this study as it was found appropriate to fit the data. The bivariate Archimedean Gumbel Copula which is applied in this study, is also known as the Gumbel-Hougard Copula. Haugh (2016), defined the Archimedean Gumbel Copula as:

$$C^{Gu}\ (u_1\ , u_2\ , \theta\ ) = \exp\left(-[(-log u_1)^{\theta} + (-log u_2)^{\theta}]^{\frac{1}{\theta}}\right), \quad \text{where, } \theta \in [1, \infty). \qquad (6)$$

The parameter θ estimates the degree of dependency. When θ = 1, independence is obtained and when $\theta \to \infty$, the Gumbel Copula converges to perfect positive dependence.

According to Naifar (2011), the Archimedean copula functions are attractive because the copula parameter is related to the tail dependence coefficient and Kendall's tau. The Archimedean Gumbel parameter (θ) and Kendall's tau (τ) are related together by the function, $\tau = \frac{\theta-1}{\theta} = 1 - \theta^{-1}$, where θ is a parameter. The Archimedean Gumbel copula's upper (λ_U) and lower (λ_L) tail dependence are estimated, respectively by the functions, $\lambda_U = 2 - 2^{-\theta}$ and $\lambda_L = 0$.

In applied statistics, engineering, insurance, economics and finance, the Gumbel copula is most frequently employed as an extreme value copula (Gumbel (1960), Embrechts, et al. (2002), and Longin and Solnik (2001)).

5. Parameter Estimation

The Archimedean Gumbel Copula is characterised by one dependence parameter that needs to be estimated. The inference for margins (IFM) approach is used to estimate the parameter.

6. Estimation of Risk

A good risk measure has certain attributes, including coherence. According to Artzner, et al. (1999), X and Y are say, two financial asset returns and a risk measure ρ(·) is coherent if the following four axioms are satisfied:

Axiom 1: Monotonicity $\rho\ (Y) \geq \rho(X)$ if $X \leq Y$

A greater expected loss requires a greater amount of capital to be held.

Axiom 2: Subadditivity $\rho\,(X + Y) \leq \rho(X) + \rho(Y)$

Combining or merging of two or more risky assets does not increase overall level of risk. This implies that the portfolio risk should be lower than (or equal to) the sum of the risk of the individual risky assets (Stanga, 2008). Therefore, Axiom 2 is the mathematical description of diversification which is applied in this study.

Axiom 3: Homogeneity For any number $k > 0, \rho\,(\alpha X) = \alpha\rho(X)$ where k is a constant positive amount. If $\alpha = 2$ say, then doubling the size of the loss situation, will double the risk.

Axiom 4: Translation Invariance $\rho\,(X + k) \leq \rho(X) - k$ for any constant of k

Adding an amount to the observed loss, then the capital required to mitigate the impact of the loss, increases by the same amount

The risk measures adopted are associated with gains and losses for the Exponential Distribution -Archimedean Gumbel Copula model are estimated. The estimation of the portfolio VaR/ES is done using the Monte Carlo simulation method for an equally weighted portfolio.

7. Portfolio Simulation

Once the models for the marginal distributions and the dependence structure (co-movement) are determined, the study simulates the unconditional loss distribution of the portfolio and estimates the investment risk measures of interest.

a. Estimation Procedure: A five-step estimation procedure (Hotta et al. 2008) is followed when fitting the Exponential distribution- Archimedean Gumbel Copula model

Step 1: Log returns are calculated and the Exponential distribution is fitted to the two data sets, viz: losses and gains separately.

Step 2: The log-return series are transformed into standard uniform (0, 1) variates and are assumed to be identically and independently distributed observations. A plot of the uniform marginals will help determine the appropriate copula for the pair of transformed data series.

Step 3: Marginal scatterplots are used to determine which bivariate copula (joint distribution) is to be fitted to the bivariate return series for each of the gains and losses separately, and estimate the parameter τ, (the Kendall's tau) in each case. θ i s then estimated using the IFM estimation method.

Step 4: Use the estimated copula model (for this bivariate data, it is found to be the Archimedean Gumbel copula is appropriate for both the losses and gains paramete r to simulate N (N = 5 000 in this research) uniform random numbers and transfor m them to the original scales of the log returns using the inverse quantile function of the joint distribution. Use the average of the input parameters as the new Exponential distribution parameters.

Step 5: Estimate the VaR and ES of the equally weighted portfolio. In this study, the portfolio weights of the two Indices were considered to be equal, but this is not a constraint and they can be varied freely (Clemente and Romano, 2005). The portfolio VaR/ES risk measures are estimated by the Monte-Carlo simulation method and then used to calculate diversification effects/benefits.

8. Diversification Effects

To capture the portfolio VaR accurately, the Exponential Distribution - Archimedean copula model is applied, and the diversification effects estimated. According to Piwcewicz (2005), Inanoglu (2007) and Yoshiba (2015), the diversification effects formula is given as:

$$\text{Diversification effects for VaR} = \frac{Simple\ Sum\ of\ VaR \quad - Aggregate\ VaR}{Simple\ Sum\ of\ \ VaR} \times\ 100\% \quad (7)$$

$$\text{Diversification effects for ES} = \frac{Simple\ Sum\ of\ \ ES \quad - Aggregate\ \ ES}{Simple\ Sum\ of\ \ ES} \times 100\% \qquad (8)$$

where Simple Sum of VaR is the total sum from the addition of the VaR values of the two risky factors; Simple sum of ES is the total sum from the addition of the ES values of the two risky factors; Aggregate VaR is the portfolio VaR of the two risky factors; and Aggregate ES is the portfolio ES of the two risky factors.

The expressions in Equation (7) and (8) were used to estimate the diversification effects of the portfolio. The study is confined to a bivariate case using an equally weighted portfolio, although in principle the ideas discussed can be applied to higher dimension models.

9. Test for Stationarity, Heteroscedasticity and Autocorrelation

To apply the Exponential Distribution - Archimedean Gumbel Copula model, the two financial data distributions are assumed to be stationary and continuous. In the following Table 1, the tests for stationarity, heteroscedasticity and autocorrelation are described.

Table 1: Test for stationarity, heteroscedasticity and autocorrelation.

Test	Method
Stationarity	The ADF test (also known as unit root or non-stationary test) is used to tests for stationarity in the South African Industrial Index (J520) and South African Financial Index (J580) return series.
Heteroscedasticity	To test for the presence of Arch effects, the Lagrange Multiplier (LM) Test is used to test for the presence of heteroscedasticity in returns of South African Industrial Index (J520) and South African Financial Index (J580) return series.
Auto-correlation	The Ljung-Box test is used to test for autocorrelation of each of the South African Industrial Index (J520) and South African Financial Index (J580) return series.

10. Test for Goodness of Fit for the Exponential Distribution

The Bartlett's test for goodness of fit is also used to test whether or not the data comes from an Exponential distribution. The hypotheses are:

H_0: the return distribution is Exponential.

H_1: the return distribution is not Exponential.

If the p-value is greater than 0.05, the null hypothesis is accepted. This would mean that there is sufficient evidence to say that the data follows an Exponential distribution.

IV. Empirical Results

This section applies the Exponential Distribution - Archimedean Gumbel copula model to the monthly South African Industrial Index (J520) returns and monthly South African Financial Index (J580) returns over the period 1995 to 2018, in analysing this bivariate portfolio and its diversification effects thereof.

1. Software used and Research Data

Data was analysed using R programming statistical packages: *actuar, Copula, fCopulae, QRM, Mass, evir, eva, fitdistrplus, fExtremes* and *extremes*. This study uses the South African Industrial and Financial Indices secondary data extracted from the website iress expert: https://expert.inetbfa.com (with permission). The analysis involved specifically the use of the South African Industrial Index (J520) and the South African Financial Index (J580) loss distributions (spanning the years: 1995–2018). These Indices are calculated for stocks of industrial sector companies and financial services sector companies listed on the South African stock market, specifically the Johannesburg Stock Exchange (JSE), and represents the performance of the specific industries in the stock market. Heyman and Santana (2017) stated that, although the South Africa's All Share Index (ALSI) is weakly efficient, the sub-Indices of the ALSI may not be informationally efficient. This may allow investors to make excess profits/losses when invested in the sub-indices which may not be informationally efficient. The sub-indices returns for the South African Industrial Index (J520) and South African Financial Index (J580) may be modelled using the Exponential distributions which is relatively positively skewed. The bivariate portfolio is constructed from the two stock Indices returns. The resultant portfolio is used to estimate the portfolio VaR/ES using the Exponential Distribution - Gumbel copula model and diversification benefits are then calculated.

The monthly logarithmic returns for both Indices are estimated as follows:

$$x_t = ln\ M_t/M_{t-1} \qquad (9)$$

where x_t represents the monthly log returns at month t, M_t is the monthly index value at month t and *ln* is the natural logarithm.

2. Exploratory Data Analysis

This section presents the descriptive data analysis and initial intuition on how the datasets are distributed. The descriptive statistics for the monthly South African Industrial Index (J520) and the monthly South African Financial Index (J580) return series are given in Table 2. The skewness is positive for both indices, which implies that the extreme values are present in the return series. The results for the kurtosis indicate that the return series exhibit excess kurtosis, implying the returns distributions have somewhat heavy-tails and exhibit leptokurtosis. This implies that the Exponential Distribution may be suitable as marginal distributions in both datasets.

a. Descriptive Statistics

Table 2: Summary of statistics.

	Industrial Index (J520)	Financial Index(J580)
No of Observations	271	271
Minimum	-0.1403	-0.2165
Maximum	0.3285	0.5119
Mean	-0.0094	-0.0083
Median	-0.0105	-0.0102
Variance	0.0033	0.0037
Skewness	1.0169	2.1943
Kurtosis	4.4209	19.4395

b. Tests for Stationarity, Heteroscedasticity and Auto-correlation

The return series data are checked for stationarity, heteroscedasticity and auto-correlation**.**

Testing for Stationarity

The Augmented Dickey Fuller (ADF) test is used to investigate whether the monthly South African Industrial Index (J520) and the monthly South African Financial Index (J580) returns are a stationary series.

Testing for Heteroscedasticity

The Arch (LM) test is applied to test for heteroscedasticity in the South African Industrial Index (J520) and the South African Financial Index (J580) return series. The Arch (LM) test checks for the presence of ARCH effects. There were no ARCH effects in the South African Industrial Index (J520) :(χ^2 = 8.3670, df = 12, p-value = 0.7558) and the South African Financial Index (J580): (χ^2 = 6.2386, df = 12, p-value = 0.9036). The p-values are greater than 0.05 for both the return series data, confirming that there are no Arch effects in both datasets.

Testing for Auto-Correlation

The Ljung-Box test is used to test for auto-correlation in the monthly South African Industrial Index (J520) and the monthly South African Financial Index (J580) return series. The results reveal a p-value > 0.05 for each of the return series data, indicating weak evidence against the null hypothesis, so we fail to reject the null hypothesis of no auto-correlation. This means that the return distribution is independently distributed. The Indices return series are each independent and identically distributed. Therefore, applying the EVT to the return series is appropriate as each series is independently and identically distributed. The two series however, may be dependent to each other, especially at the extremities. This is discussed in the next section.

c. Parameter Estimates

The parameters are obtained from Exponential distribution fit of the two financial asset return distributions are shown in Table 3. These parameters are used to estimate the VaR and ES of the two financial assets for both univariate and bivariate distributions.

Table 3: Estimates Exponential distribution

Financial Asset	**Parameter Estimate($\hat{\lambda}$)**
Left Tail of the loss distribution/Gains /upper tail	
J520	24.2566
J580	23.6812
Right Tail of the loss distribution/Losses/lower tail	
J520	22.4897
J580	24.2560

4. Diagnostic Tests for J520 and J580 losses

Graphical techniques are given in Figure 1 and 2 to assess the goodness of fit of the Exponential distribution for losses only.

a) J520 Exponential distribution losses

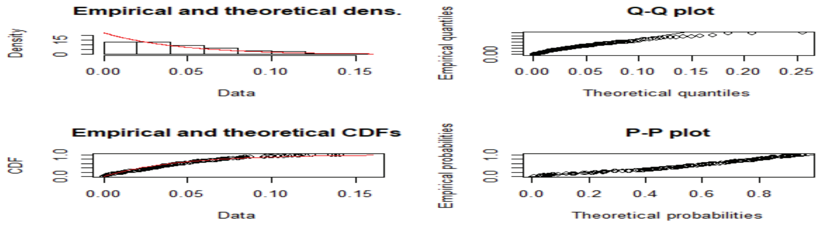

Figure 1: Diagnostic plots for the J520 Exponential distribution losses.

b**) J580 Exponential distribution Losses**

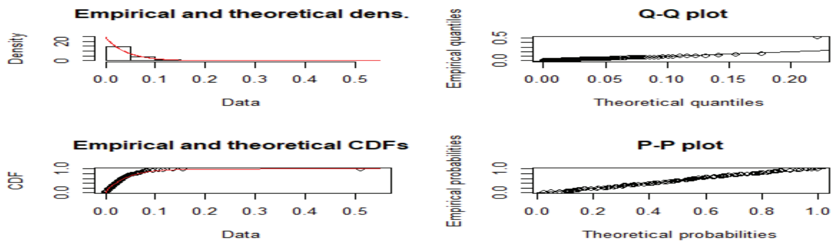

Figure 2: Diagnostic plots for the J580 Exponential distribution loss returns.

The histograms and theoretical density coincide as shown in Figure 1 and 2. The data points exhibit linearity in the P-P and Q-Q plots, and show insignificant deviation from the 45°line in the Figures. From the diagnostic plots, the Exponential distribution is a good fit for the data. The diagnostic plots for the Exponential gains were done but not presented in paper. They also show that the Exponential distribution is a good fit to the data.

Bartlett's test of goodness of fit for the Exponential Distribution

In Table 4, all the p-values are greater than 0.05, therefore the null hypothesis of Exponentially distributed data is accepted.

Table 4: p-values for Bartlett's Test for Exponential distribution.

Index	J520 p-values	J580 p-values
Losses	0.2370	0.1861
Gains	0.0604	0.1116

Selection of appropriate copula function for bivariate analysis

The scatterplots in Figure 3 and Figure 4 are plotted in order to determine the type of copula to fit to the bivariate gains and the bivariate losses. The gains are separated from the losses for each of the series. Losses are negative returns multiplied by negative one to make them positive.

In Figure 3, it is inconclusive which copula to propose for analysis of the bivariate gains. It appears that there is little tail dependence. The Archimedean Gumbel copula is therefore suggested since it is able to model extreme values under varying degrees of dependence. Increasing the dependence parameter increases the degree of upper tail dependence in the Gumbel copula. Lower values imply little or no upper tail dependency. The is no lower tail dependence for a Gumbel copula. The interest is in extreme risk in the upper right-hand corner of the figure.

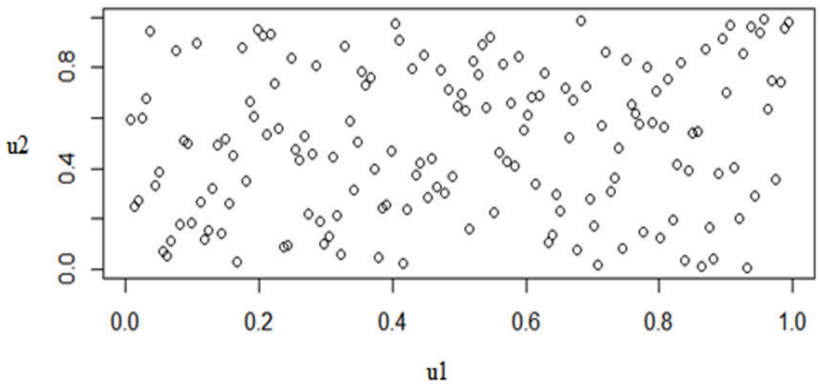

Figure 3: Scatterplot of Exponential distribution marginals u_1 and u_2 for the bivariate gains.

In Figure 4, the bivariate series shows a very weak increasing pattern and converging in the upper right corner, suggesting an upper tail dependence, and so again the Archimedean Gumbel copula is suggested.

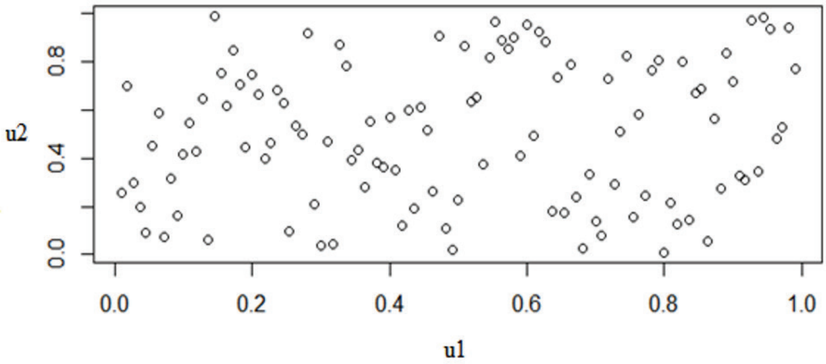

Figure 4: Scatterplot of Exponential distribution marginals u_1 and u_2 for the bivariate losses.

Kendall's tau, copula parameter, upper and lower tail dependence measures

In Table 5, the estimated Archimedean Gumbel copula parameters (θ) for the gains and losses are 1.1113 and 1.0900 respectively. The Gumbel copula parameters imply the presence of weak upper tail dependence in the gains and losses. This means that large gains and losses from the two stock indices have some small chance to co-move together concurrently (Embrechts et al, 2001). This implies that the two stock market indices may tend to rise and fall together during periods of economic recession, though weakly so. The upper tail dependence measures for gains and losses are 0.2713 and 0.1114 respectively. Tail dependence measures indicate the degree of extreme co-movements of large gains/losses in the stock markets which allows investors and practitioners to estimate portfolio risk and calculate diversification effects.

Table 5: Archimedean Gumbel copula: Kendall's tau, Copula Parameter, upper and lower tail dependence measures for the gains and losses.

Family	Kendal's tau $\hat{\tau}$	Copula Parameter $\hat{\theta}$	Upper tail $\widehat{\lambda^U}$	Lower tail $\widehat{\lambda^L}$
Gumbel copula for gains	0.5922	1.1113	0.2713	0
Gumbel copula for losses	0.4554	1.0900	0.1113	0

Archimedean Gumbel copula density and contour plots

The Archimedean copulas allows modelling dependence in high dimensions with only one parameter, governing the strength of dependence. In Figure 5, the density plots for the Gumbel copula for both the gains and the losses are shown while in Figure 6, the contour plots confirm the presence of some upper tail dependence for the gains and losses respectively. The Archimedean Gumbel copula is characterized by the presence of upper tail dependence as can be concluded from the density and contour plots respectively.

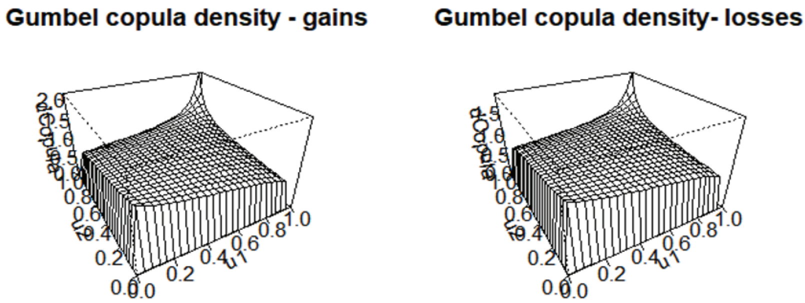

Figure 5: Density plots of the joint distribution for the Gumbel copula gains (θ=1,1113) and Gumbel copula losses (θ=1.0900).

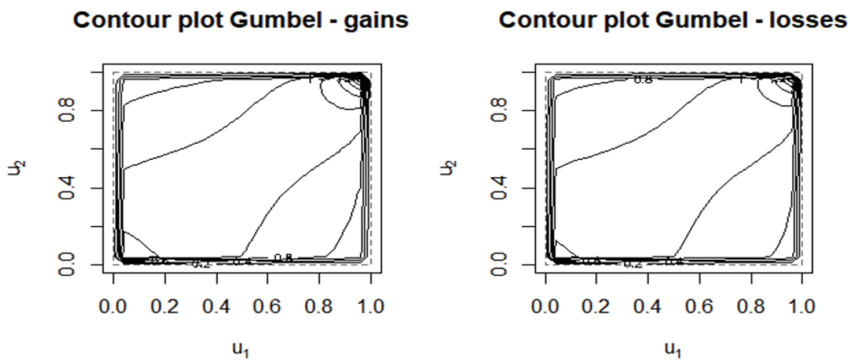

Figure 6: Contour plots of the joint distribution for the Archimedean Gumbel copula gains (θ=1,1113) and Archimedean Gumbel copula losses (θ=1.0900).

Estimation of univariate risk measures

In Table 6, the univariate risk measures for the two financial assets are estimated using the Exponential distribution model.

Table 6: Univariate risk measures for J520 and J580 using Exponential distribution.

	J520		**J580**	
Alpha	VaR	ES	VaR	ES
Measures of Risk-Left tail of loss distribution (gains)				
0.950	0.1365 (0.0006)	0.1582 (0.0006)	0.1120 (0.0010)	0.1463 (0.0010)
0.990	0.2164 (0.0006)	0.2292 (0.0006)	0.1628 (0.0010)	0.2010 (0.0010)
0.995	0.2527 (0.0006)	0.2586 (0.0006)	0.1928 (0.0010)	0.2317 (0.0010)
Measures of Risk-Right tail of loss distribution (losses)				
0.950	0.1168 (0.0006)	0.1556 (0.0006)	0.1265 (0.0010)	0.1623 (0.0010)
0.990	0.1669 (0.0006)	0.2095 (0.0006)	0.1945 (0.0010)	0.2548 (0.0010)
0.995	0.1875 (0.0006)	0.2308 (0.0006)	0.2237 (0.0010)	0.2648 (0.0010)

In Table 6 the gains and losses are used to estimate the univariate VaR and ES (their standard errors are in brackets) of the South African Industrial Index (J520) and the South African Financial Index (J580). For VaR and ES, the J580 losses are riskier than the J520 losses since the risk measures are bigger. However, for VaR and ES, the J520 gains are greater than the J580 gains risk measures. These risk measures are used to determine the diversification effects of the portfolio.

Estimation of portfolio risk using the Exponential Distribution -Archimedean Gumbel copula model

Table 7 summarises the average parameters of J520 and J580 (see Table 3) used in determination of the inverse Copula distribution.

Table 7: Average parameters for Inverse Copula Distribution.

Average parameter $\hat{\lambda}$ for gains	23.9689
Average parameter $\hat{\lambda}$ for losses	23.3728

Table 8 summarizes the estimated VaR and ES for the portfolio. For the gains at higher quantiles, with a 95 % level of confidence, the Exponential distribution - Archimedean Gumbel copula gives VaR and ES estimates of 0.2018 and 0.2683 respectively. The results are interpreted as follows: the expected market gains for the portfolio will not go above 0.2018 at the 95% confidence level, if it goes beyond, it will average 0.2683. The interpretation is the same for all the other estimates. The estimated portfolio risk in a bivariate setting can be used to account for the diversification effects. This study therefore provides investors with a tool which allows them to assess and avoid extreme risks and at the same time, benefit from extreme gains.

Table 8: Estimation of portfolio risk using Exponential distribution - Archimedean copula model.

Copula	Marginals	**portfolio VaR**			**portfolio ES**		
		95 %	**99%**	**99.5 %**	**95 %**	**99%**	**99.5 %**
Gains							
Gumbel	Exponential	0.2018	0.3004	0.3489	0.2683	0.3841	0.4474
Losses							
Gumbel	Exponential	0.2052	0.3028	0.3503	0.2718	0.3845	0.4456

Estimation of diversification effects

In Table 9, at 95% level of confidence for the gains, the portfolio incurs diversification effects of 49.40%. For the losses at 95% level of confidence, the portfolio incurs diversification effects of 51.89%. There is slightly greater diversification benefits in losses for the portfolio than the gains, which is favourable for the risk averse investor. On the other hand, in Table 10 at 95% level of confidence, for the gains, the portfolio incurs diversification effects of 48.33 %. For the losses at 95% level of confidence, the portfolio incurs diversification effects of 48.45%. The ES is a coherent risk measure and the diversification benefits for gains and losses are more in line with each other. For investors these findings provide meaningful implications for diversification decisions used to hedge against exposure to risk.

Table 9: Estimation of diversification effects using VaR.

Alpha	VaR for J520	VaR for J580	Simple Sum for VaR	VaR for Portfolio	Diversification Effects
Exponential Distribution -Gumbel copula: gains					
95	0.1235	0.1265	0.2500	0.1265	49.40%
99	0.1898	0.1944	0.3842	0.1945	49.45%
99.5	0.2184	0.2337	0.4521	0.2237	48.30%
Exponential Distribution -Gumbel copula: losses					
95	0.1332	0.1235	0.2567	0.1235	51.89%
99	0.2048	0.1898	0.3946	0.1899	51.87%
99.5	0.2356	0.2184	0.4540	0.2184	51.89%

Table 10: Estimation of diversification effects using the ES.

Alpha	ES for J520	ES for J580	Simple Sum for ES	ES for Portfolio	Diversification Effects
Exponential Distribution - Gumbel Copula: Gains					
95	0.1582	0.1621	0.3203	0.1655	48.33%
99	0.2291	0.2347	0.4638	0.2342	49.51%
99.5	0.2585	0.2648	0.5233	0.2636	49.64%
Exponential Distribution - Gumbel copula: Losses					
95	0.1707	0.1582	0.3289	0.1695	48.45%
99	0.2472	0.2291	0.4763	0.2397	49.69%
99.5	0.2789	0.2585	0.5374	0.2680	50.14%

V. CONCLUSION/ DISCUSSION

The Exponential distribution-Archimedean Gumbel copula model is preferable to the traditional, correlation-based approach as the model is able to capture non-linear dependencies. The portfolio diversification results show a decrease in portfolio risk when compared to the total sum of the risk of the two single risky assets. This implies that the portfolio offers diversification benefits which point to reduction in losses for investors holding the portfolio. The implication is that, investors who invested in an individual risky asset with a VaR /ES higher than the portfolio may consider adding the financial risky assets to form the portfolio in order to reduce exposure to risk.

The diversification effects/benefits estimated are consistent with results estimated by Inanoglu (2007), Shaw and Spivak (2009) and Yoshiba (2016) when the analysis was done to those that range up to 60% in diversification benefits. The diversification effects/benefits and risk measures presented provided a better understanding of the level of interdependencies among different industries/stock markets. Estimation of portfolio risk and diversification effects/benefits offers investors and practitioners information to identify the current opportunities and risks in holding the portfolio.

For future possible research, the authors are interested in estimating diversification effects using the EVT- copula model which uses the more specialised GPD and GEVD marginals to characterise the very extreme returns in the tails of distributions.

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